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Lecture Series:

Structural Dynamics

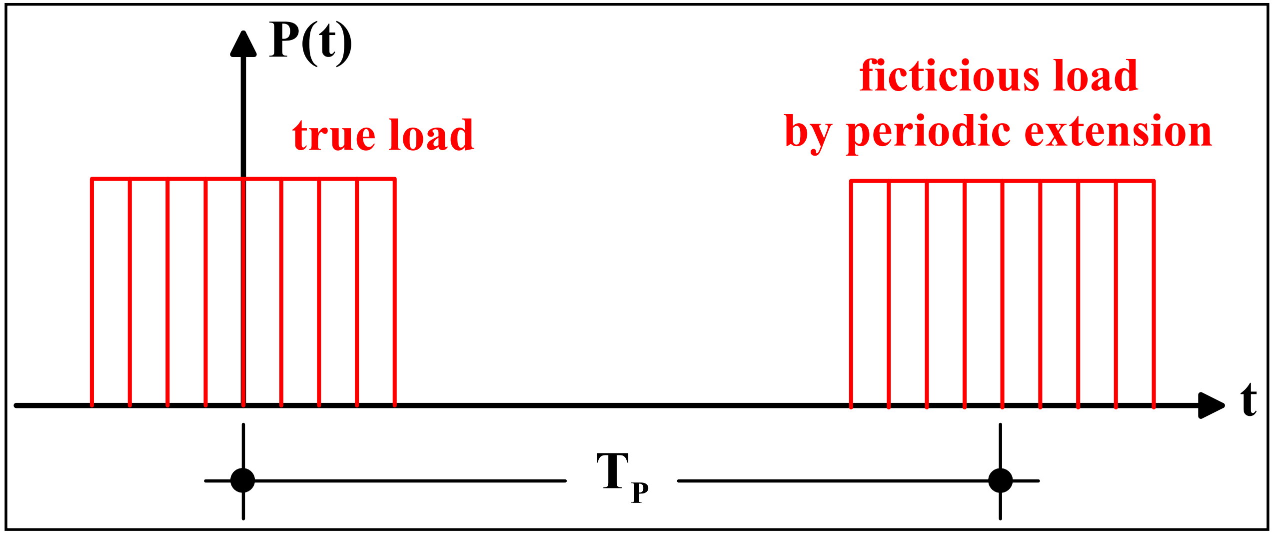
Mathematical Review 2:

Harmonic Analysis

Part B: Fourier Transformation

Problem with Aperiodic Functions: Periodic Mirroring of Function

We have seen that any *periodic function* can be expressed by a *FOURIER series*, i.e. an infinite sum of *harmonic functions*. The *periodicity* is part of the formulation: the sine and cosine functions are periodic by nature, and so any sum of them is per force periodic, too. We could, however, add zeros after a non-periodic signal to achieve a kind of quasi-nonperiodicity. Still, after T_P time units fictitious mirror images would be created. If, however, we would let T_P approach infinity ($T_P \rightarrow \infty$), then the influence of the periodic extensions converges to zero. The FOURIER series becomes the *FOURIER transformation*.



Starting Point: Fourier Series

We start with the FOURIER series in sine/cosine formulation:

$$P(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\Omega_n t) + \sum_{n=1}^{\infty} b_n \sin(\Omega_n t)$$

$$a_0 = \frac{2}{T_P} \int_{-T_P/2}^{T_P/2} P(t)\, dt$$

$$a_n = \frac{2}{T_P} \int_{-T_P/2}^{T_P/2} P(t) \cos \Omega_n t \, dt$$

$$b_n = \frac{2}{T_P} \int_{-T_P/2}^{T_P/2} P(t) \sin \Omega_n t \, dt$$

$$\Omega_P = \frac{2\pi}{T_P}$$

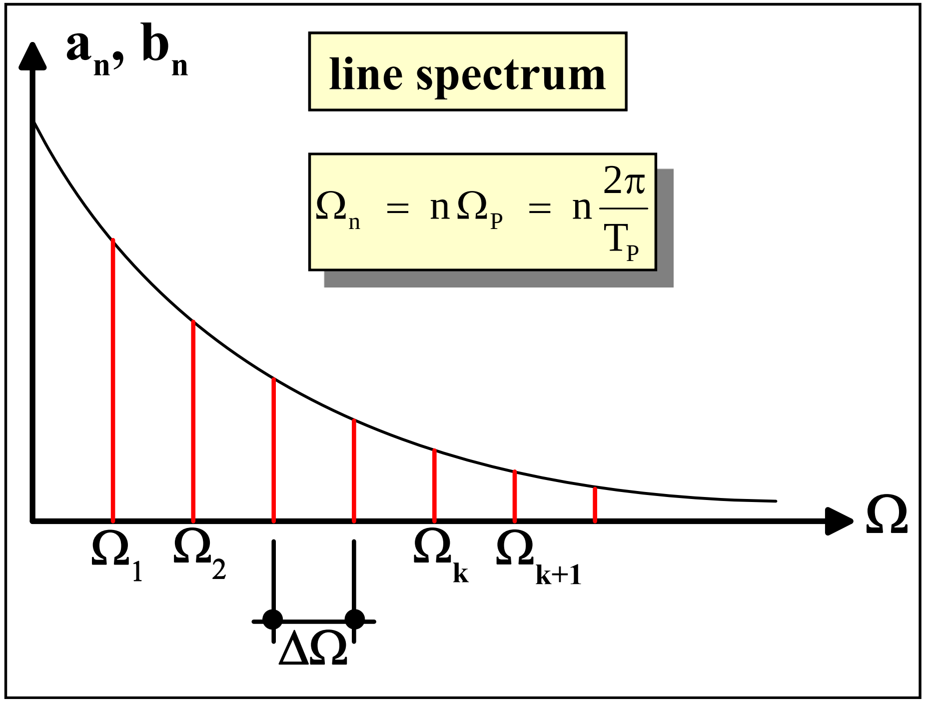
$$\Omega_n = n\,\Omega_P$$

Now we extend the sum to also encompass the term for n = 0, i.e. $\Omega_0 = 0$:

$$P(t) = \sum_{n=0}^{\infty} [a_n \cos(\Omega_n t) + b_n \sin(\Omega_n t)] - \frac{a_0}{2}$$

Definition of the Frequency Increment

The *spectrum* of the FOURIER coefficients is a *line spectrum*: we have values for a_n and b_n at discrete frequencies Ω_n. The spectrum is not defined between these discrete points. The frequencies Ω_n are spaced equidistantly with a *constant frequency increment $\Delta\Omega$*.



$$\Omega_n = n\Omega_P = n\frac{2\pi}{T_P}$$

$$\Delta\Omega = \Omega_{k+1} - \Omega_k$$

$$\Delta\Omega = (k+1)\Omega_P - k\Omega_P = \Omega_P = \frac{2\pi}{T_P}$$

$$\frac{2}{T_P} = \frac{\Delta\Omega}{\pi}$$

Introduction of the Frequency Increment

FOURIER formula:

$$P(t) = \sum_{n=0}^{\infty} [a_n \cos(\Omega_n t) + b_n \sin(\Omega_n t)] - \frac{a_0}{2}$$

We substitute a_n and b_n by the FOURIER integrals where we substitute $2/T_P$ by $\Delta\Omega$:

$$P(t) = \sum_{n=0}^{\infty} \left[\left\{ \int_{-T_P/2}^{T_P/2} \frac{\Delta\Omega}{\pi} P(\tau) \cos\Omega_n \tau \, d\tau \right\} \cos\Omega_n t + \left\{ \int_{-T_P/2}^{T_P/2} \frac{\Delta\Omega}{\pi} P(\tau) \sin\Omega_n \tau \, d\tau \right\} \sin\Omega_n t \right] - \int_{-T_P/2}^{T_P/2} \frac{\Delta\Omega}{2\pi} P(\tau) \, d\tau$$

Now we let T_P approach infinity. As T_P becomes infinitely large, so $\Delta\Omega$ becomes infinitely small, i.e it becomes a differential quantity: $\Delta\Omega$ becomes $d\Omega$. The formerly discrete spectrum becomes a *continuous spectrum*. The discrete sum over all Ω becomes an integral over all frequencies between zero and infinity.

The constant term in the series for P(t) vanishes since the integral over the time domain gives a finite value which is multiplied by the quantity $d\Omega$ which is infinitely small:

$\Delta\Omega \to d\Omega$ ⇒

$$\int_{-T_P/2}^{T_P/2} \frac{\Delta\Omega}{2\pi} P(\tau) \, d\tau \to \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(\tau) \, d\tau \, d\Omega = 0$$

From the FOURIER Sum to the FOURIER Integral

The frequency increment $\Delta\Omega$ does not depend on time and can be extracted from the integrals:

$$P(t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\left[\left\{\int_{-T_P/2}^{T_p/2} P(\tau)\cos\Omega_n\tau\,d\tau\right\}\cos\Omega_n t\Delta\Omega + \left\{\int_{-T_P/2}^{T_p/2} P(\tau)\sin\Omega_n\tau\,d\tau\right\}\sin\Omega_n t\Delta\Omega\right]$$

The sum becomes an integral in the limit case where $\Delta\Omega$ becomes $d\Omega$ (T_p approaches infinity). The discrete FOURIER coefficients a_n and b_n are now continuous functions since the gaps in the discrete line spectrum shrink to zero.

$$P(t) = \frac{1}{\pi}\int_0^{\infty}\left[\left\{\int_{-\infty}^{+\infty} P(\tau)\cos\Omega\tau\,d\tau\right\}\cos\Omega t + \left\{\int_{-\infty}^{+\infty} P(\tau)\sin\Omega\tau\,d\tau\right\}\sin\Omega t\right]d\Omega$$

We re-introduce the now continuous FOURIER coeffecients:

$$P(t) = \frac{1}{\pi}\int_0^{\infty}\left[a(\Omega)\cos\Omega t + b(\Omega)\sin\Omega t\right]d\Omega$$

FOURIER Transformation

spectral domain → time domain (FOURIER synthesis)

$$P(t) = \frac{1}{\pi}\int_{0}^{\infty}\left[a(\Omega)\cos\Omega t + b(\Omega)\sin\Omega t\right]d\Omega$$

time domain → spectral domain (FOURIER decomposition)

$$a(\Omega) = \int_{-\infty}^{+\infty} P(t)\cos\Omega t\,dt$$

$$b(\Omega) = \int_{-\infty}^{+\infty} P(t)\sin\Omega t\,dt$$

[a(Ω), b(Ω)]: FOURIER transform of the original function P(t).

Complex FOURIER Transformation

spectral domain → time domain

$$P(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{c}_P(\Omega)\, e^{i\Omega t}\, d\Omega$$

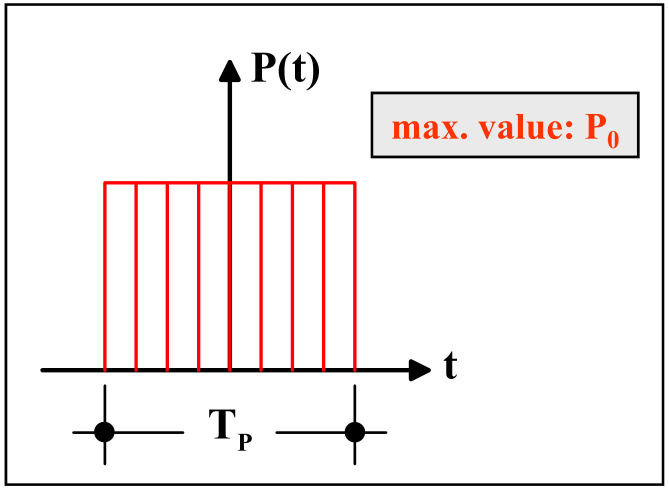
time domain → spectral domain

$$\underline{c}_P(\Omega) = \int_{-\infty}^{+\infty} P(t)\, e^{-i\Omega t}\, dt$$

$$\underline{c}_P(\Omega) = a(\Omega) - i\, b(\Omega)$$

$\underline{c}_P(\Omega)$: FOURIER transform of the original function P(t)

Example: Rectangular Impulse



Analytical determination of the FOURIER transform:

$$\underline{c}_P(\Omega) = \int_{-\infty}^{+\infty} P(t)\, e^{-i\Omega t}\, dt = \int_{-T_P/2}^{+T_P/2} P_0\, e^{-i\Omega t}\, dt$$

$$\underline{c}_P(\Omega) = P_0 \left[-\frac{1}{i\Omega} e^{-i\Omega t} \right]_{-T_P/2}^{+T_P/2} = \frac{P_0\, i}{\Omega} \left\{ e^{-i\Omega T_P/2} - e^{i\Omega T_P/2} \right\}$$

$$e^{i\varphi} = \cos\varphi + i \sin\varphi$$

$$\underline{c}_P(\Omega) = \frac{P_0\, i}{\Omega} \left\{ -2i \sin(\Omega T_P / 2) \right\}$$

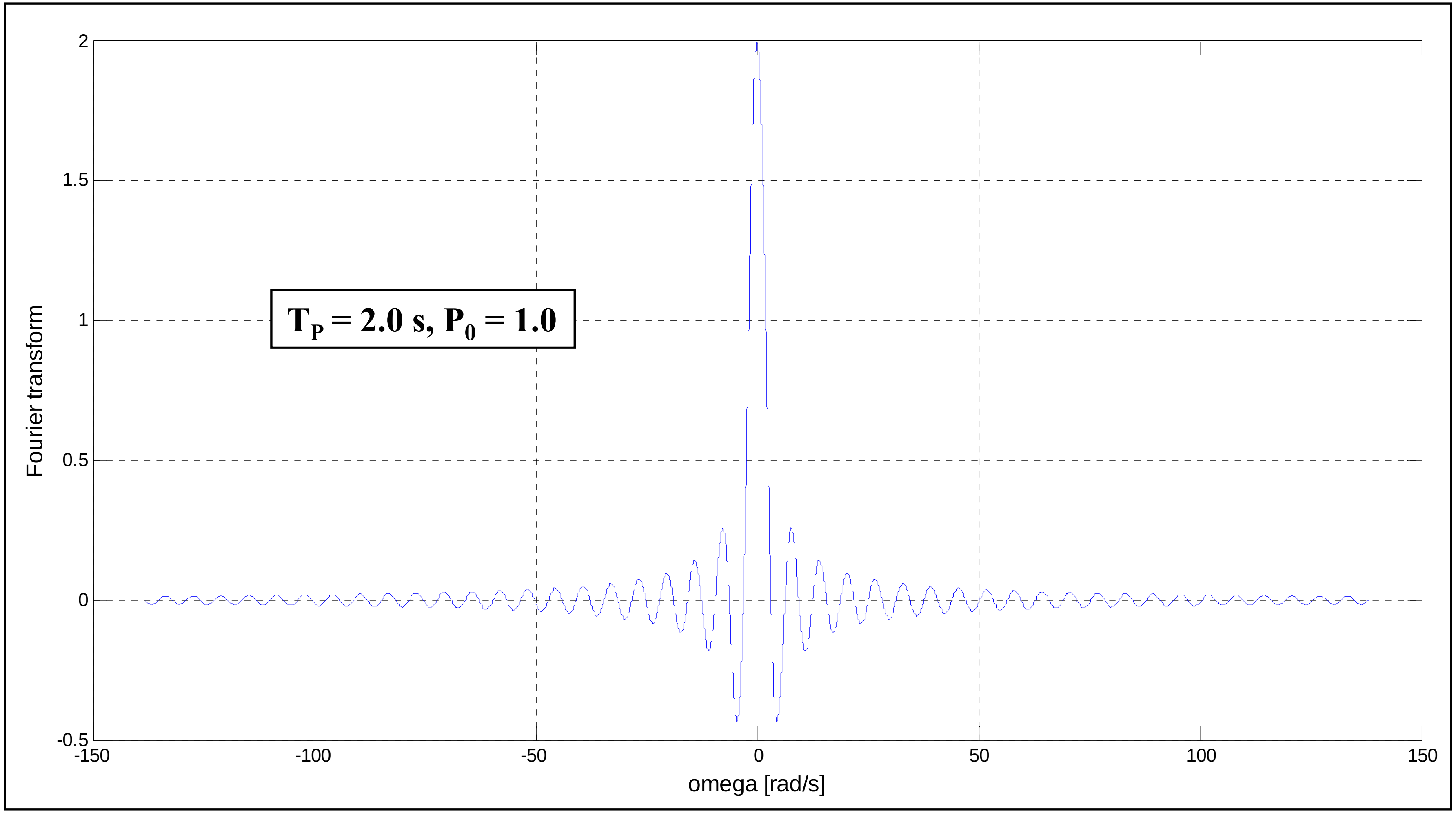
$$\underline{c}_P(\Omega) = \frac{2 P_0}{\Omega} \sin(\Omega T_P / 2)$$

not defined for $\Omega = 0$

special case for $\Omega = 0$

$$\underline{c}_P(0) = P_0 T_P$$

FT for the Rectangular Impulse



Inverse Transformation into the Time Domain

inverse FOURIER transformation

$$P(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{c}_P(\Omega)\, e^{i\Omega t}\, d\Omega$$

FOURIER transform

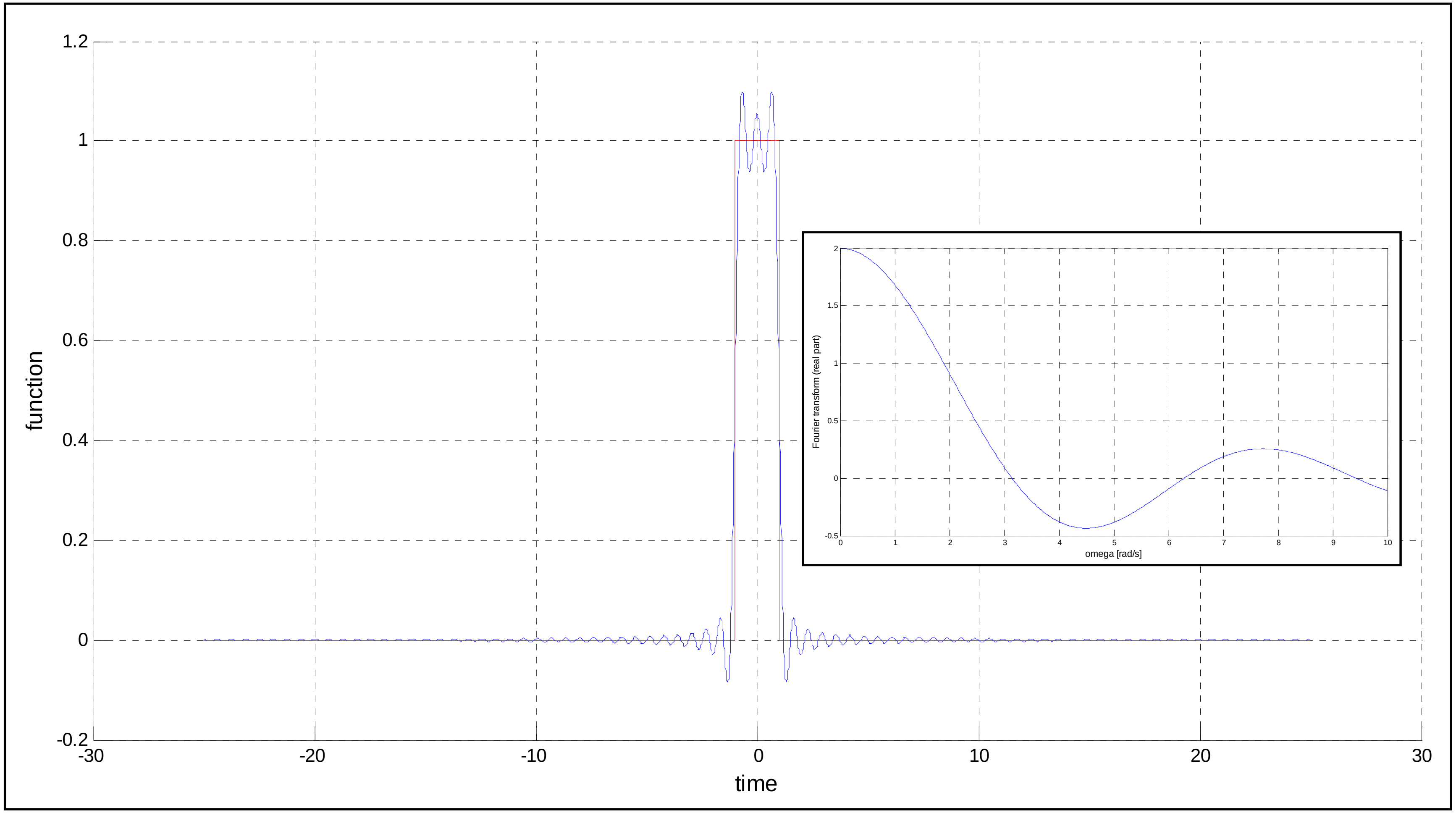
$$\underline{c}_P(\Omega) = \frac{2P_0}{\Omega}\sin(\Omega T_P/2)$$

The inverse transformation can only be done numerically since there is no analytical expression for the original function which is valid for the whole time domain. We choose a fixed value for t and calculate the integral over the 'entire' frequency range, choose another value for t and repeat the numerical integration. Each data point of the time signal requires the evaluation of an integral. We have set 'entire' in inverted commas since the integration algorithm requires a numerical value for the integration boundary. This value can be large, but it is still finite. We cannot integrate over the entire frequency range but only over a range [$-\Omega_{max}$: $+\Omega_{max}$] which we consider relevant.

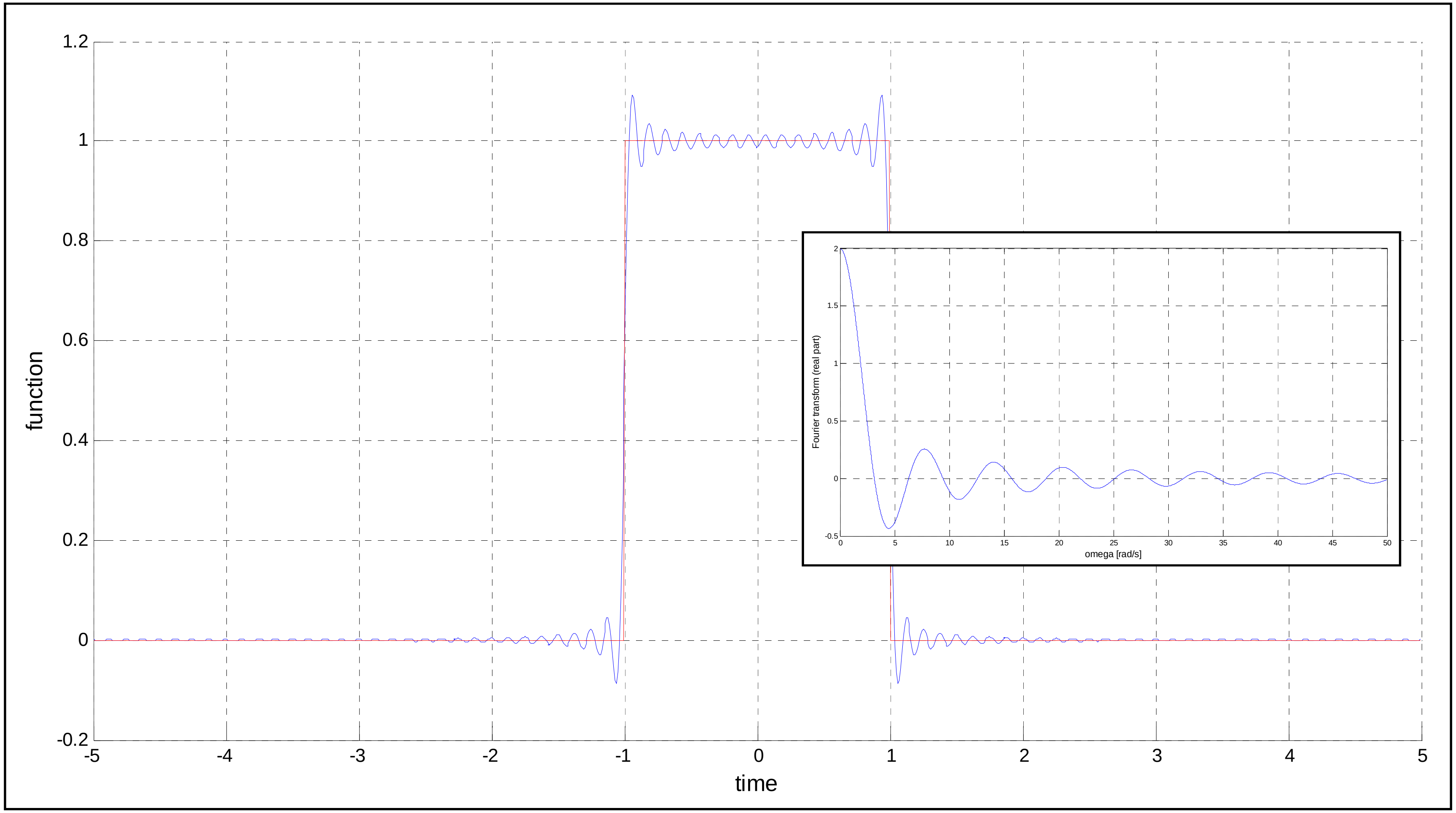
The quality of the solution is determined by 3 parameters:

- **the integration algorithm (here: trapezoidal rule),**
- **the integration boundaries, and**
- **the size of the frequency steps.**

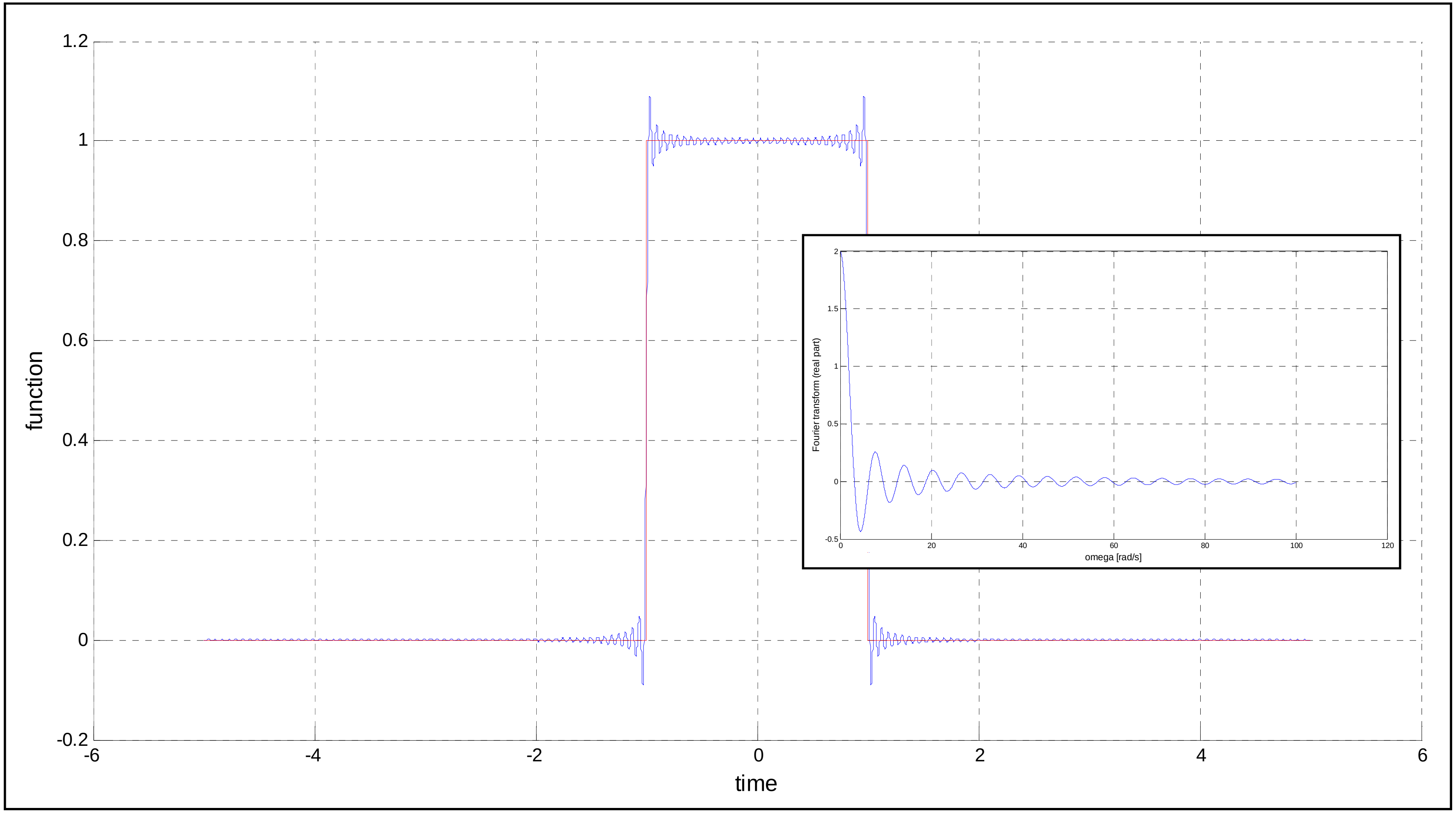
$\Omega_{max} = 10.0$ rad/s, $\Delta\Omega = 0.01$ rad/s



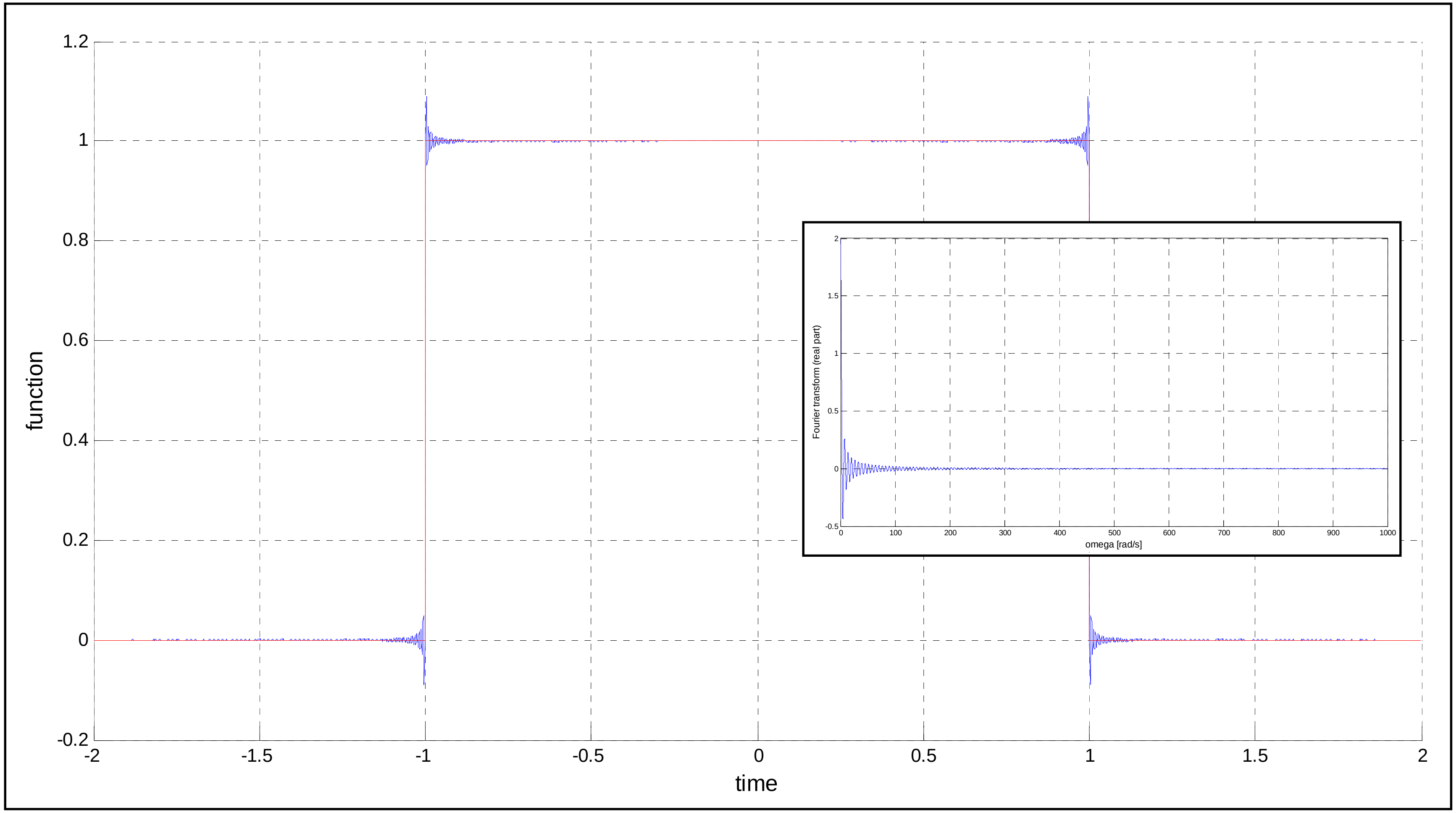
Ω_{max} = 50.0 rad/s, $\Delta\Omega$ = 0.01 rad/s



$\Omega_{max} = 100.0$ rad/s, $\Delta\Omega = 0.01$ rad/s



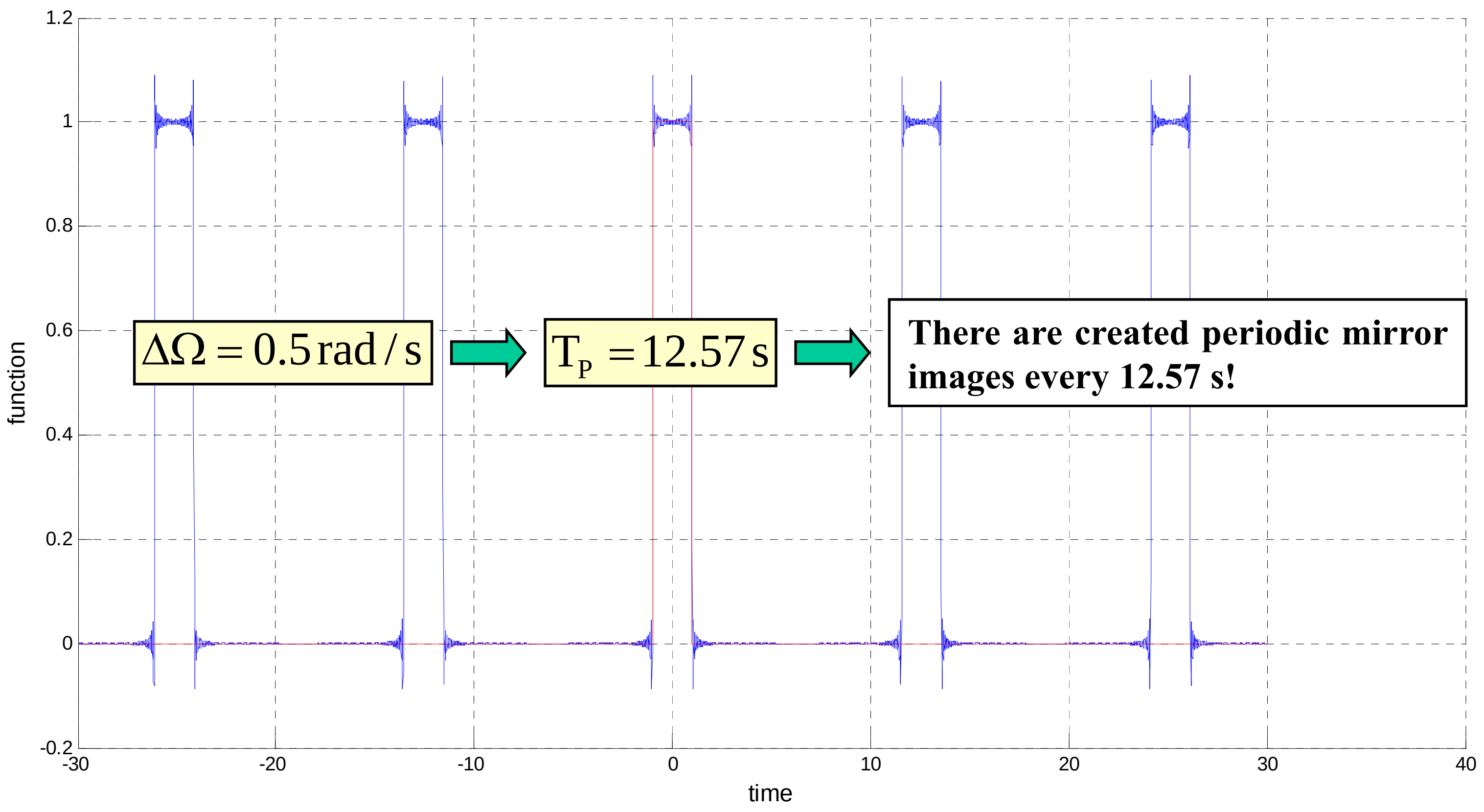
Ω_{max} = 1000.0 rad/s, $\Delta\Omega$ = 0.01 rad/s



Finite Frequency Increment $\Delta\Omega$

We have integrated the frequency range with a finite frequency step $\Delta\Omega$. The frequency increment is directly related to the period T_P so that a finite $\Delta\Omega$ automatically implies also a finite T_P.

$$\Delta\Omega = \frac{2\pi}{T_P} \quad \Rightarrow \quad T_P = \frac{2\pi}{\Delta\Omega}$$



$\Delta\Omega = 0.5\,\mathrm{rad/s}$ → $T_P = 12.57\,\mathrm{s}$ → **There are created periodic mirror images every 12.57 s!**

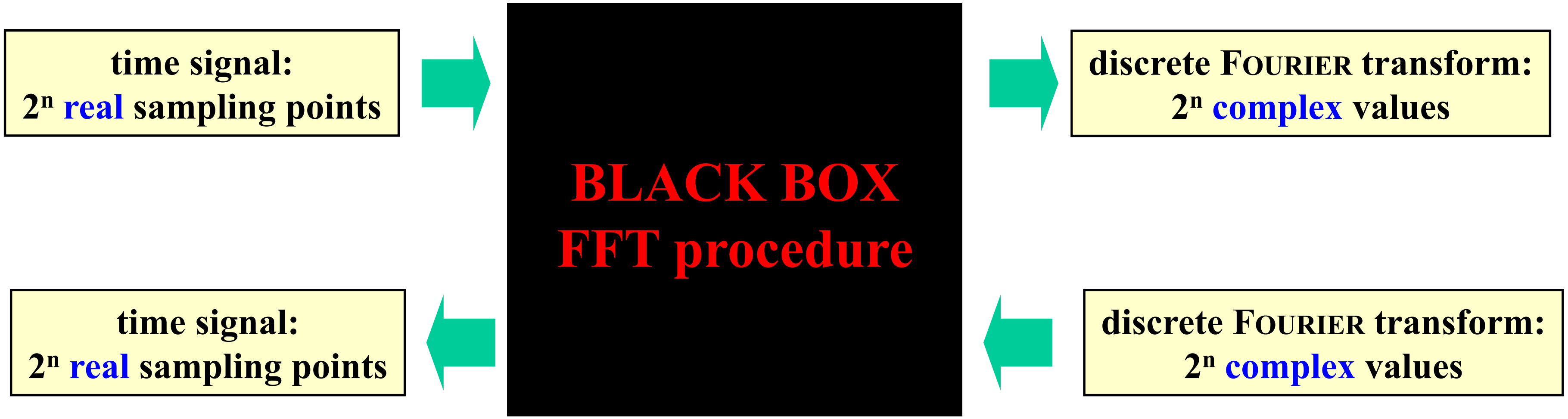
FFT: FAST FOURIER TRANSFORMATION

A calculation of the FOURIER decomposition or synthesis via numerical integration is extremely time-consuming as we have seen by running our little test programs. In 1965 there was published a landmark paper on the numerical computation of FOURIER series by a special algorithm, the ***FFT*** or ***FAST FOURIER TRANSFORMATION***, which is incredibly fast compared to numerical integration. This paper is one of the most important papers ever published in numerical mathematics.

J.W. COOLEY & J.W. TUKEY 1965:
"An Algorithm for Machine Calculation of Complex Fourier Series"

- A ***recursive algorithm*** for the FOURIER transformation of **discrete samples**.
- The **FOURIER transform** is approximated by a **discrete FOURIER series**.
- The number of sample points must be a **power of 2**.
- Same algorithm works for FOURIER decomposition (FFT) and synthesis (IFFT).
- Time signal is periodically repeated after the sample length.
- FOURIER transform is mirrored with respect to the **NYQUIST frequency**.
- Algorithm not readily understandable by the average user – black box procedure.

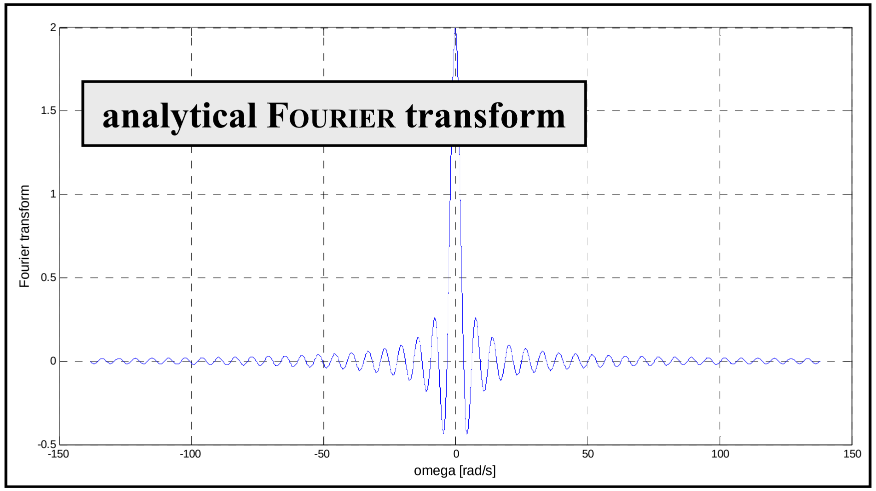
FFT: Black Box Application

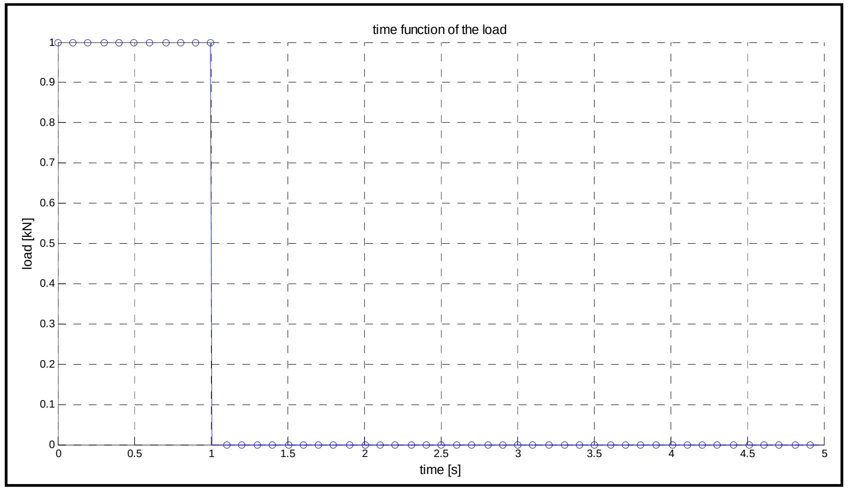


We shove a sample of *2^n real input values* into the FFT box and get *2^n complex output values* out of it. A transformation, however, cannot create information, it can only change the form of information. So if we have an original information content of 2^n values (one real value for each time instance), we cannot possibly get an information package of size $2 \cdot 2^n$ (two values for each complex frequency instance). Only half of the information of the discrete FOURIER transform can be independent information, the other half must be implied in the first half.

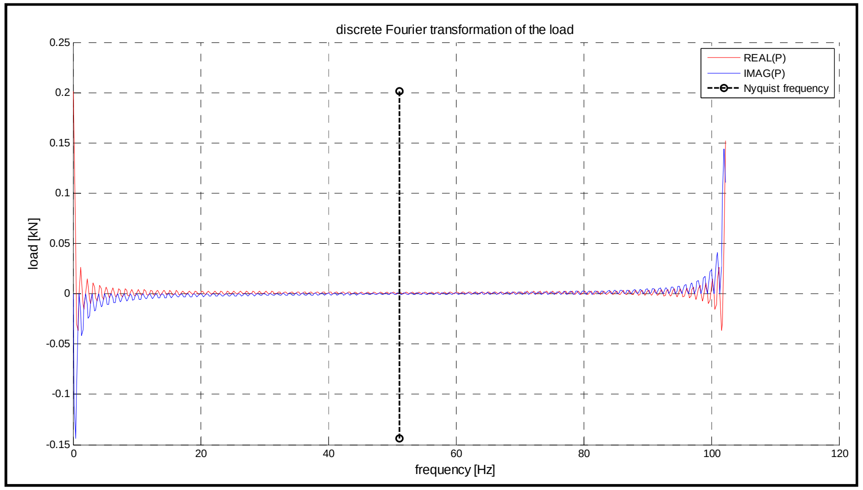
The Nyquist Frequency

The analytical Fourier transform on the right shows a convergence to zero as the frequency approaches infinity. When performing an FFT, however, we will find that the Fourier coefficients are mirrored with respect to a frequency that lies in the middle between f_{max} and f_{min}. This mirror frequency is called Nyquist frequency. The Fourier coefficients up to the Nyquist frequency are true values, while the mirror values are numerical artefacts of the FFT which must be disregarded. The IFFT, however, requires the presence of the mirror values – otherwise the synthesized time signal would not be real but complex.

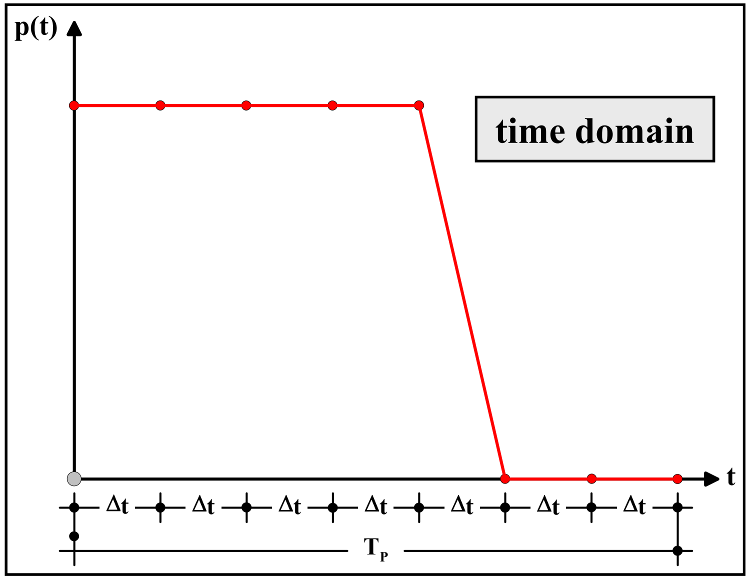


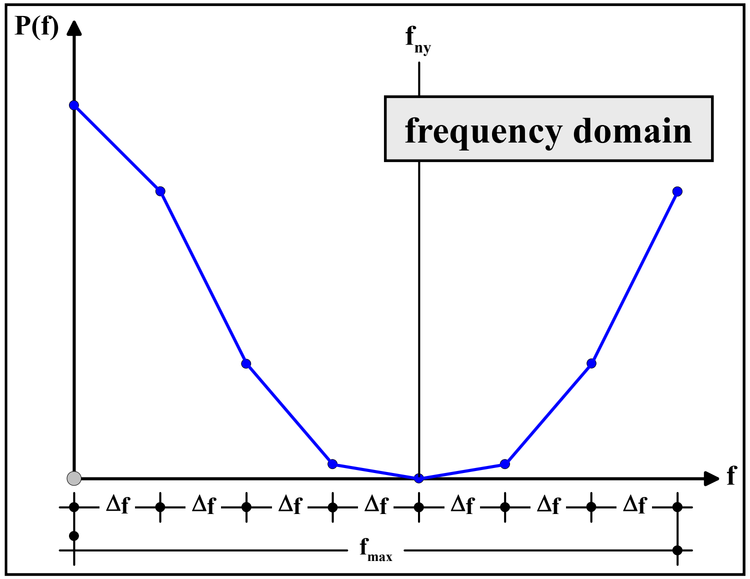


FFT



Time/Frequency Resolution





time domain | **frequency domain**

T_{sample} → $$\Delta f_{sample} = \frac{1}{T_{sample}}$$

The duration T_{sample} of the time signal determines the frequency resolution Δf_{sample}.

$$\Delta t_{sample} = \frac{T_{sample}}{N_{sample} - 1}$$ → $$f_{sample} = \frac{1}{\Delta t_{sample}}$$

The time resolution Δt_{sample} determines the maximum frequency f_{sample} (sampling frequency).

Numerical Study

test signal:

$$p(t) = e^{-\alpha t} \sum_k A_k \sin(2\pi f_k t)$$

$$\alpha = -\frac{\ln \varepsilon}{T_\varepsilon}$$

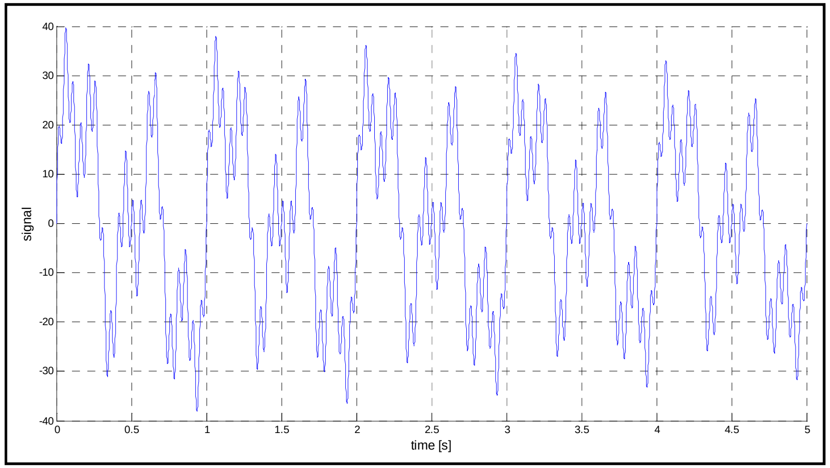
$$\varepsilon = 0.01$$

$$T_\varepsilon = 100\,s$$

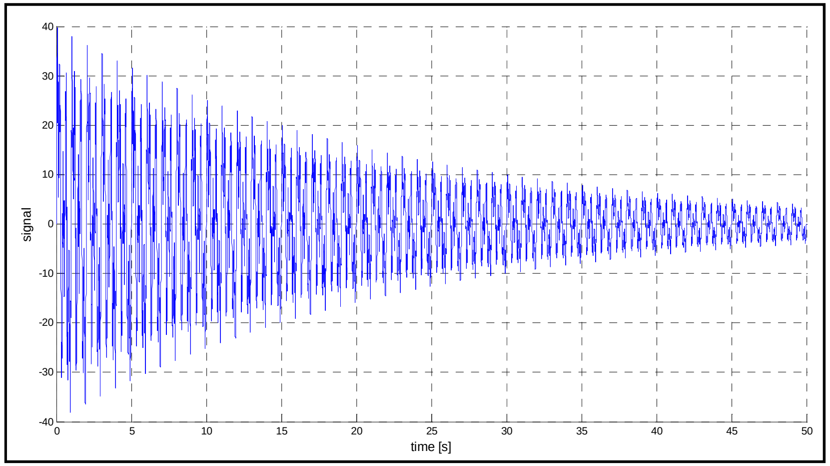
signal parameters:

k	1	2	3	4
A_k [-]	10	20	15	8
f_k [Hz]	1	2	5	20

test signal for t < 5 s:



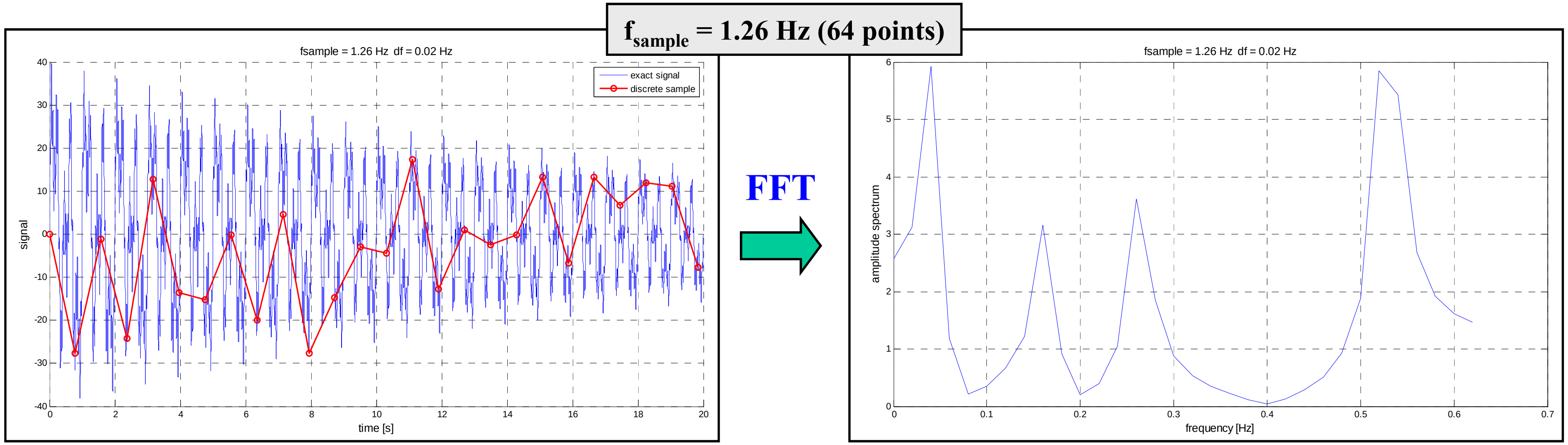
test signal for t < 50 s:



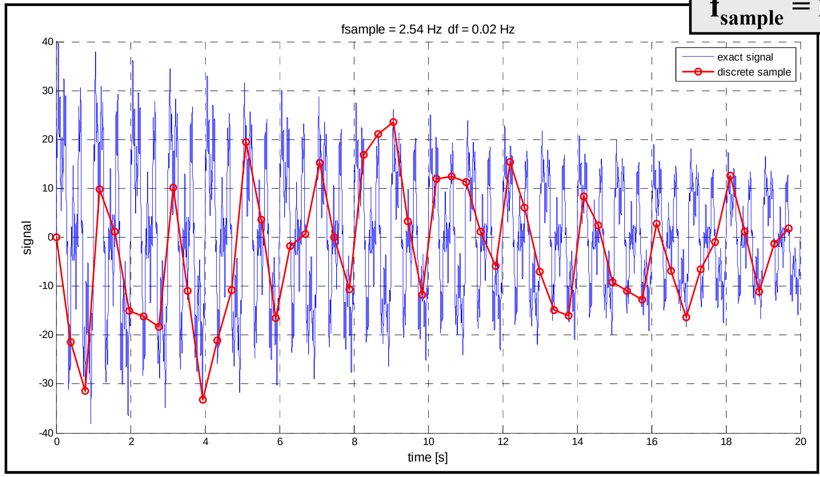
Test Program

The signal contains ***four harmonics*** with frequencies of **1 Hz, 2 Hz, 5 Hz, and 20 Hz**. We expect the FOURIER transform to show distinct peaks at exactly these frequencies and no further peaks at any other frequency. To avoid the complex nature of the FOURIER transform we plot the absolute value of the FOURIER coefficients which is equivalent to plotting the ***discrete amplitude spectrum***.

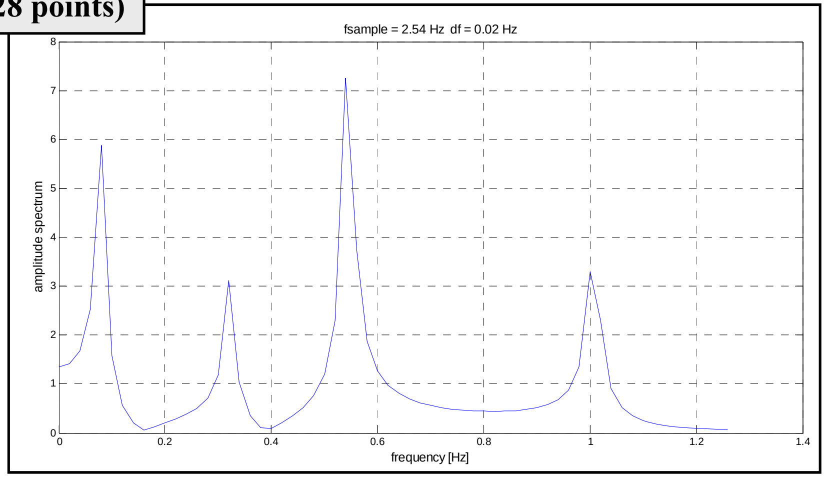
We set the sample length to $T_{sample} = 50.0$ s which yields a frequency resolution of $\Delta f = 0.02$ Hz. Then we vary the number of sample points, i.e. the sampling frequency. We start with a low number of points and study the effect of the sampling frequency on the amplitude spectrum.



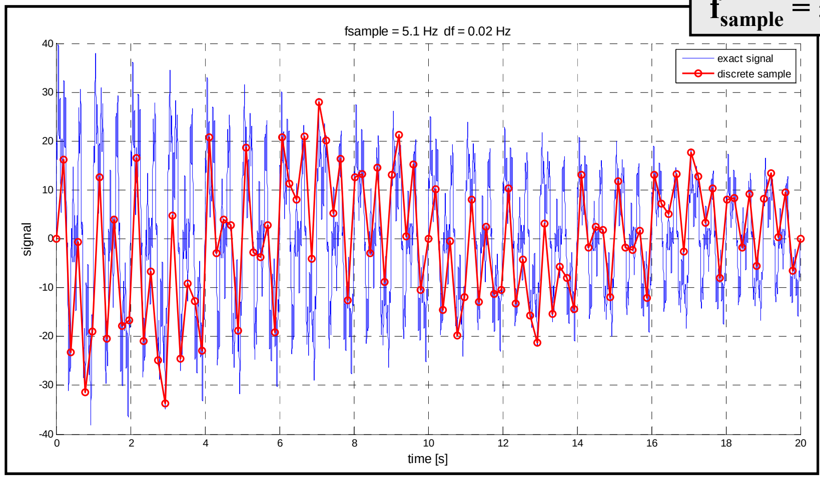
f_{sample} = 2.54 Hz (128 points)



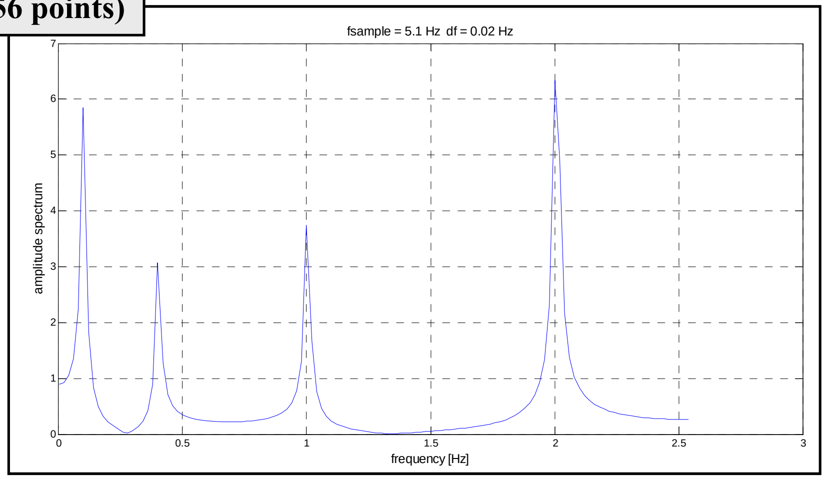
FFT



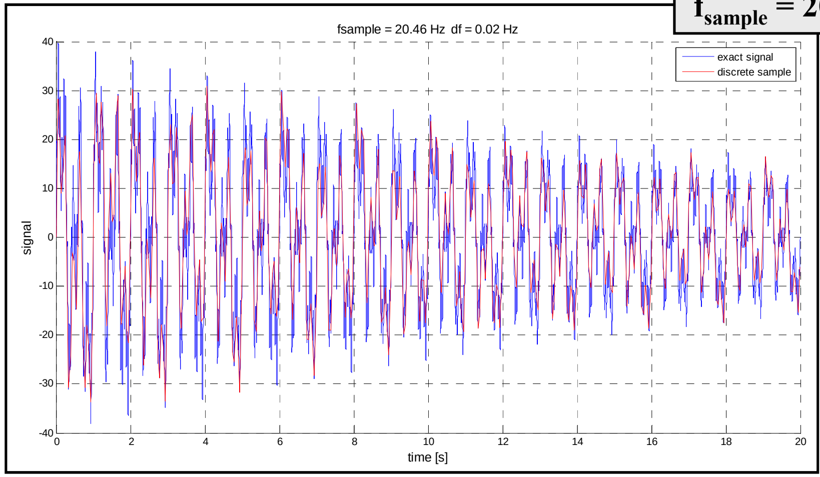
f_{sample} = 5.10 Hz (256 points)



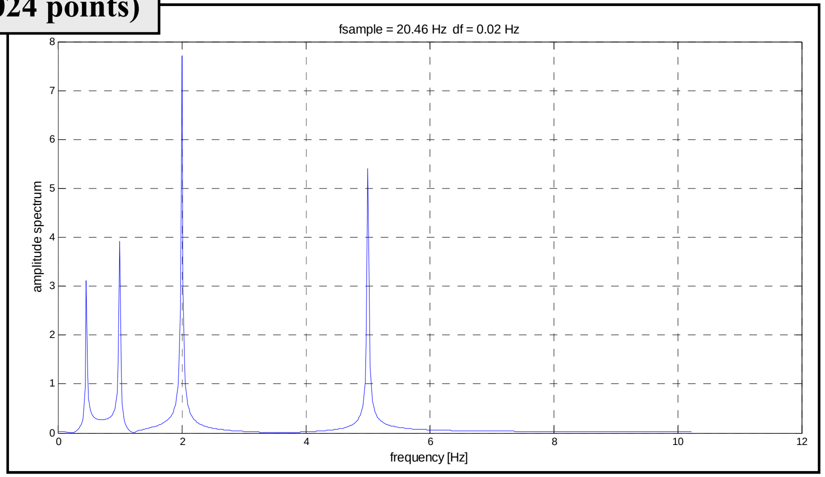
FFT



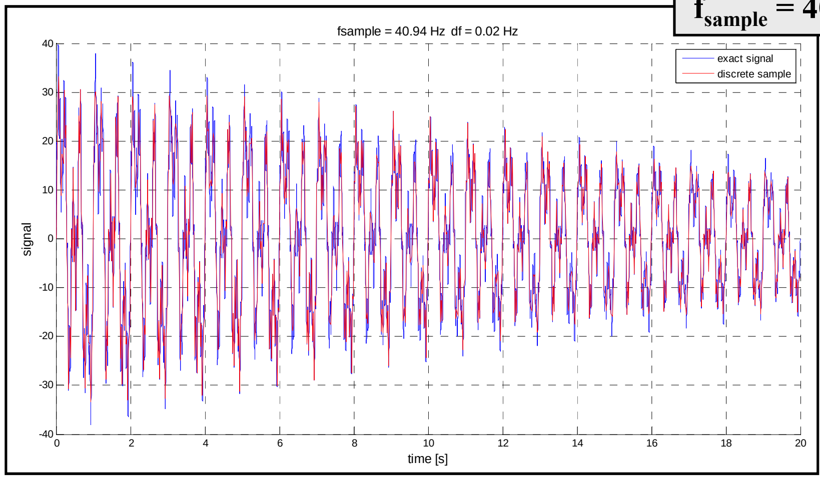
f_{sample} = 20.46 Hz (1024 points)



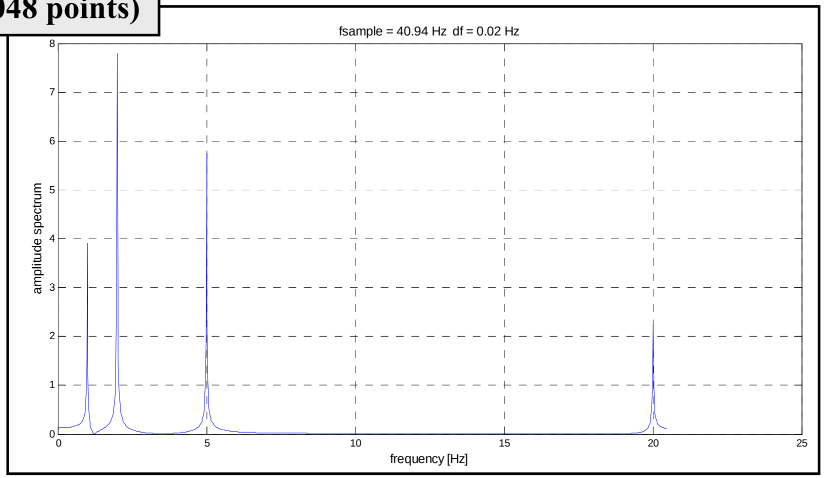
FFT



f_{sample} = 40.94 Hz (2048 points)



FFT



Discussion

When we sample the true time signal with only few points we create a numerical sample which bears little resemblance with the original signal. In particular the sample does not contain the harmonics of higher frequencies. Instead we can observe certain low-frequency periodicities which are not present in the original data but rather artefacts of the sampling process. These artificial harmonics are captured in the amplitude spectrum as peaks.

With only 64 sampling points we get 4 peaks at [0.04 | 0.16 | 0.26 | 0.52] Hz. All these peaks are artificial. The true harmonics cannot be captured since their frequencies lie beyond the NYQUIST frequency. The artificial harmonics are mirror images of the true harmonics which are mirrored from the high-frequency range into the low-frequency range.

An increase of N_{sample} to 128 leads to a halving of the time increment (well, not exactly, but close enough for our purposes here) and a doubling of the NYQUIST frequency. The peaks are shifted to [0.08 | 0.32 | 0.54 | 1.00] Hz. The last peak at 1.00 Hz is a true harmonic since now the NYQUIST frequency is 1.26 Hz. A further increase of N_{sample} again changes the artificial harmonics but not the true ones. With 2048 sampling points we have a NYQUIST frequency of about 20.5 Hz so that all true harmonics are captured correctly. A further increase of N_{sample} is not necessary: it would not change the amplitude spectrum below 20 Hz.

We see that the sampling frequency must be chosen high enough to avoid creating artificial harmonics in the spectrum which would be erroneously interpreted as for instance eigenfrequencies. More on this Topic in Lecture 14 “Experimental Techniques in Structural Dynamics”.

Summary

The *Fourier transform FT* is an extension of the *Fourier series* to *non-periodic signals*. With the FT we transform a time signal from the *time domain TD* into the *frequency domain FD*. The FT is of fundamental importance for structural dynamics. We will use this technique in the following fields:

Computation of time responses. In Lecture 9 we will develop an algorithm which yields the time domain response via a prior transformation into the FD and later re-transformation from the FD back into the TD. In the FD we are able to capture frequency-dependent properties such as frequency-dependent damping or stiffness.

Analysis in the spectral domain. For stochastic loads it is often not possible to define loads as time functions. Instead loads are given in the spectral domain by so-called auto-spectra and cross-spectra. Then the entire analysis proceeds in the FD; we can extract from this solution certain statistic properties of the structural response. More in Lecture 12.

Experimental structural dynamics. Experiments are often the only way to determine dynamic properties such as damping and eigenfrequencies since we have only limited knowledge regarding the structural properties such as member stiffness and elasticity of the supports. In these cases we measure first a time signal from which we must extract the desired dynamic properties. The extraction process is performed in the FD so that the FT is an indispensible tool when performing dynamic structural testing.