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Lecture Series:

Structural Dynamics

Mathematical Review 2:

Harmonic Analysis

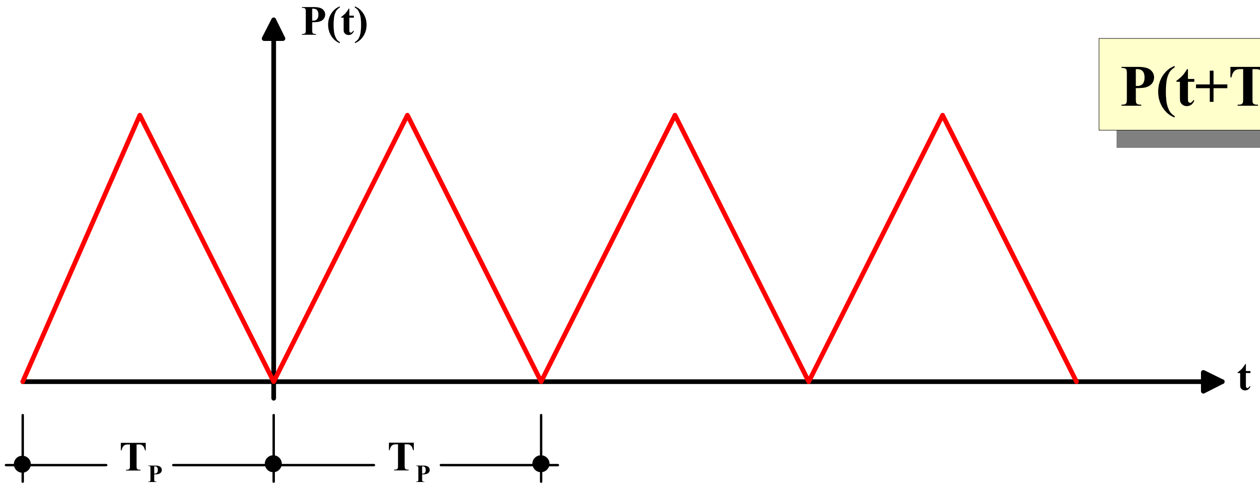
Part a: Fourier Series

Fourier Series: General Idea

Jean Baptiste Joseph Fourier (* March 21, 1768 near Auxerre; † May 16, 1880 in Paris) found out: Any *periodic function* can be expressed as an *infinite series* of *harmonic components*:

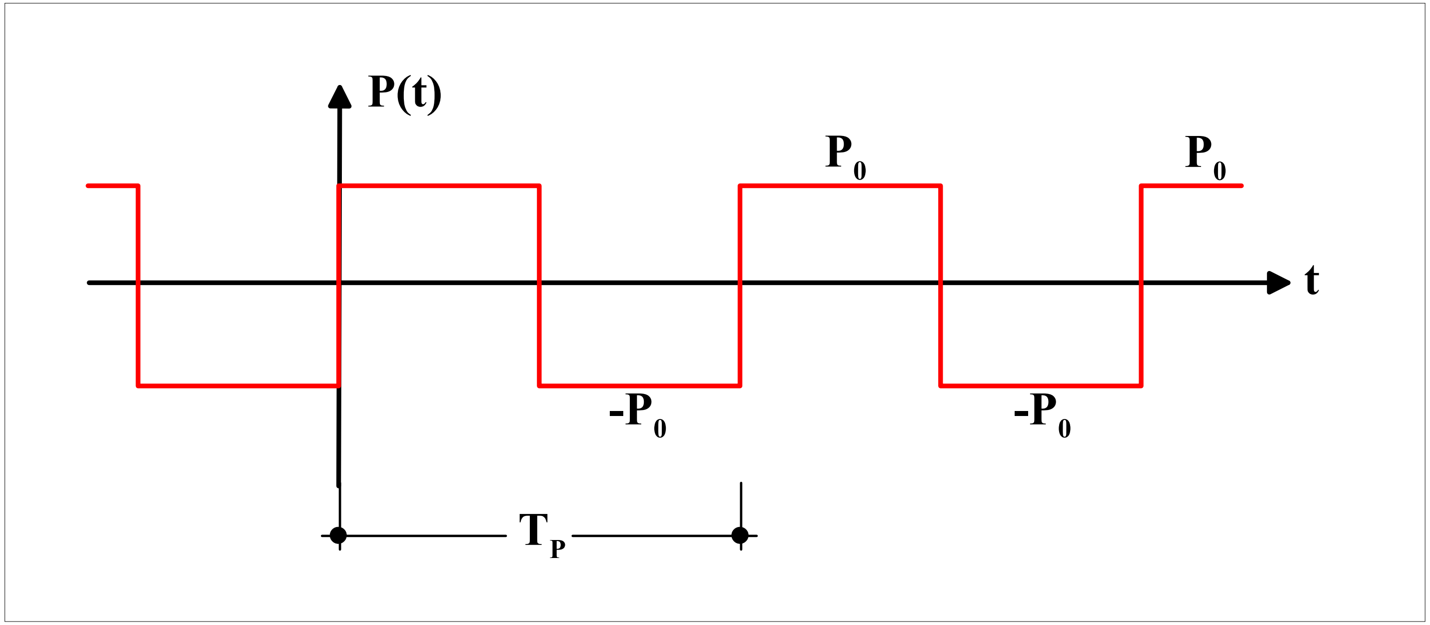
$$P(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\Omega_n t) + \sum_{n=1}^{\infty} b_n \sin(\Omega_n t)$$

A periodic function repeats its values after a certain time interval, its period T_P.



$$P(t+T_P) = P(t)$$

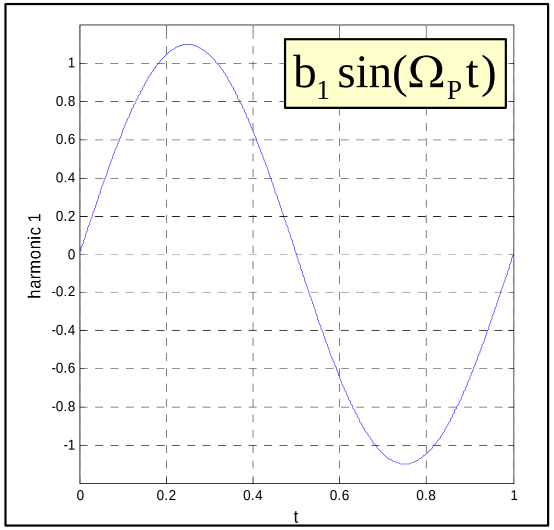
Fourier Series: Visualization



$$\Omega_P = \frac{2\pi}{T_P}$$

$$\Omega_n = n\,\Omega_P$$

$P(t) =$



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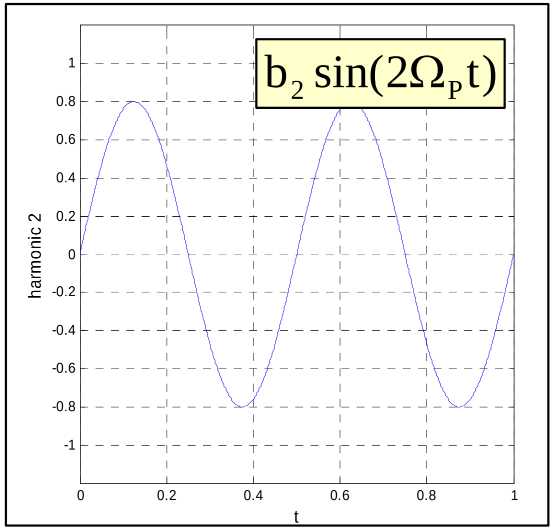

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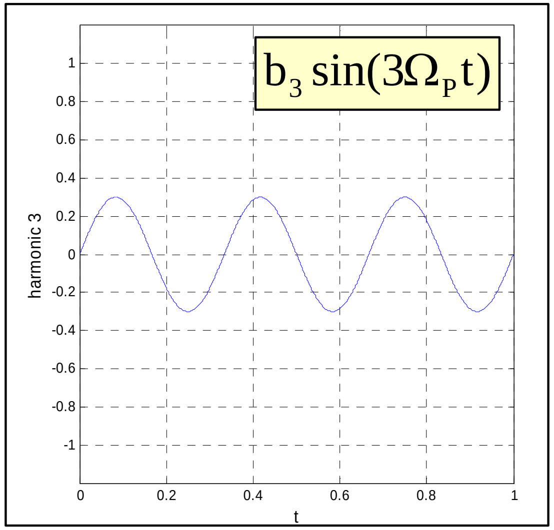


$+ \ldots$

Fourier Series: Proof I

Starting point: Fourier series (we dispense with the suffix $()_P$ for simplicity's sake

$$P(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega t)$$

We multiply by sin(mΩt)

$$P(t)\sin(m\Omega t) = \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\Omega t) + \sum_{n=1}^{\infty} b_n \sin(n\Omega t) \right\} \sin(m\Omega t)$$

We integrate over the entire period T

$$\int_{-T/2}^{T/2} P(t)\sin(m\Omega t)dt = \int_{-T/2}^{T/2} \frac{a_0}{2}\sin(m\Omega t)dt + \sum_{n=1}^{\infty} \int_{-T/2}^{T/2} a_n \cos(n\Omega t)\sin(m\Omega t)dt + \sum_{n=1}^{\infty} \int_{-T/2}^{T/2} b_n \sin(n\Omega t)\sin(m\Omega t)dt$$

Fourier Series: Proof II

The integral over an antisymmetric argument function is zero!

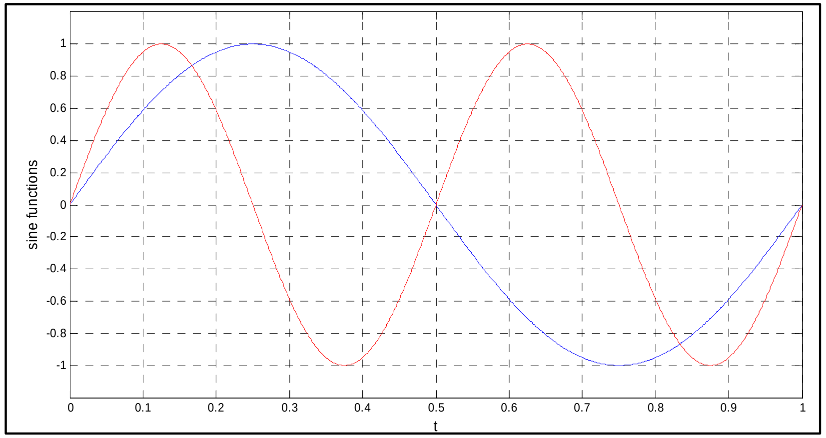
$$\int_{-T/2}^{T/2} P(t)\sin(m\Omega t)dt = \sum_{n=1}^{\infty} \int_{-T/2}^{T/2} b_n \sin(n\Omega t)\sin(m\Omega t)dt$$

The integral over a product of sine functions with different wave numbers is zero! Only the summand where n=m remains of the entire sum over n.

$$\int_{-T/2}^{T/2} P(t)\sin(m\Omega t)dt = \int_{-T/2}^{T/2} b_m \sin(m\Omega t)\sin(m\Omega t)dt = \frac{1}{2} b_m T$$

$$b_m = \frac{2}{T}\int_{-T/2}^{T/2} P(t)\sin(m\Omega t)dt$$

The expression for a_0 and a_m can be found in a similar manner. That means: the coefficients calculated by these formulas allow an approximation of any periodic function with *arbitrary accuracy*. The original function becomes identical to the Fourier series in the theoretical limit of an infinite number of harmonics.



Determination of Fourier Coefficients

General periodic function:

$$a_0 = \frac{2}{T_P} \int_{-T_P/2}^{T_P/2} P(t)\, dt$$

$$a_n = \frac{2}{T_P} \int_{-T_P/2}^{T_P/2} P(t) \cos(n\Omega_P t)\, dt$$

$$b_n = \frac{2}{T_P} \int_{-T_P/2}^{T_P/2} P(t) \sin(n\Omega_P t)\, dt$$

Symmetric periodic function: the sine parts are zero

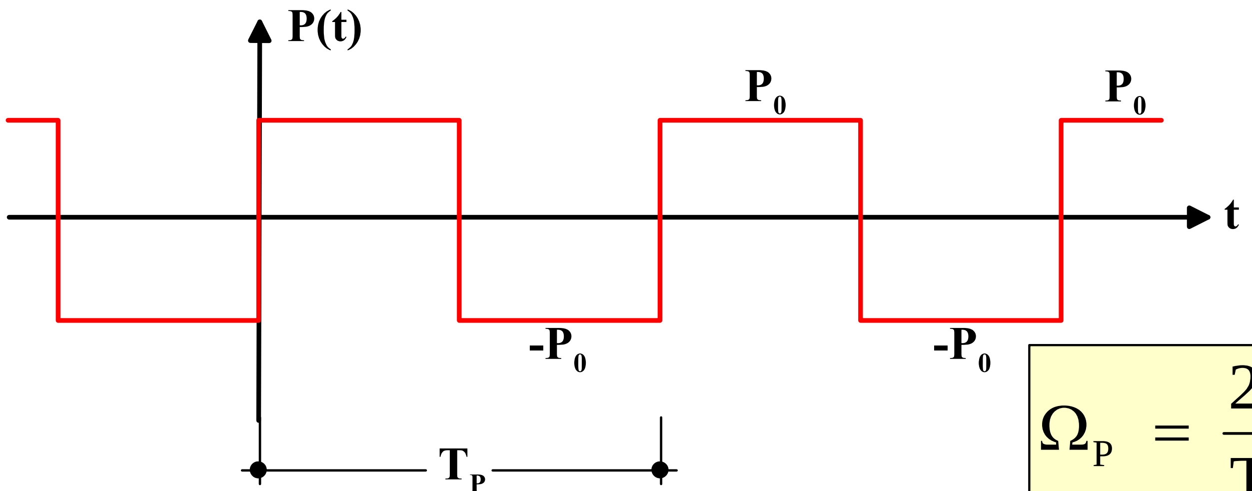
$$a_0 = \frac{4}{T_P} \int_{0}^{T_P/2} P(t)\, dt$$

$$a_n = \frac{4}{T_P} \int_{0}^{T_P/2} P(t) \cos(n\Omega_P t)\, dt$$

Antisymmetric periodic function: the cosine parts are zero

$$b_n = \frac{4}{T_P} \int_{0}^{T_P/2} P(t) \sin(n\Omega_P t)\, dt$$

Example I: Rectangular Sawtooth

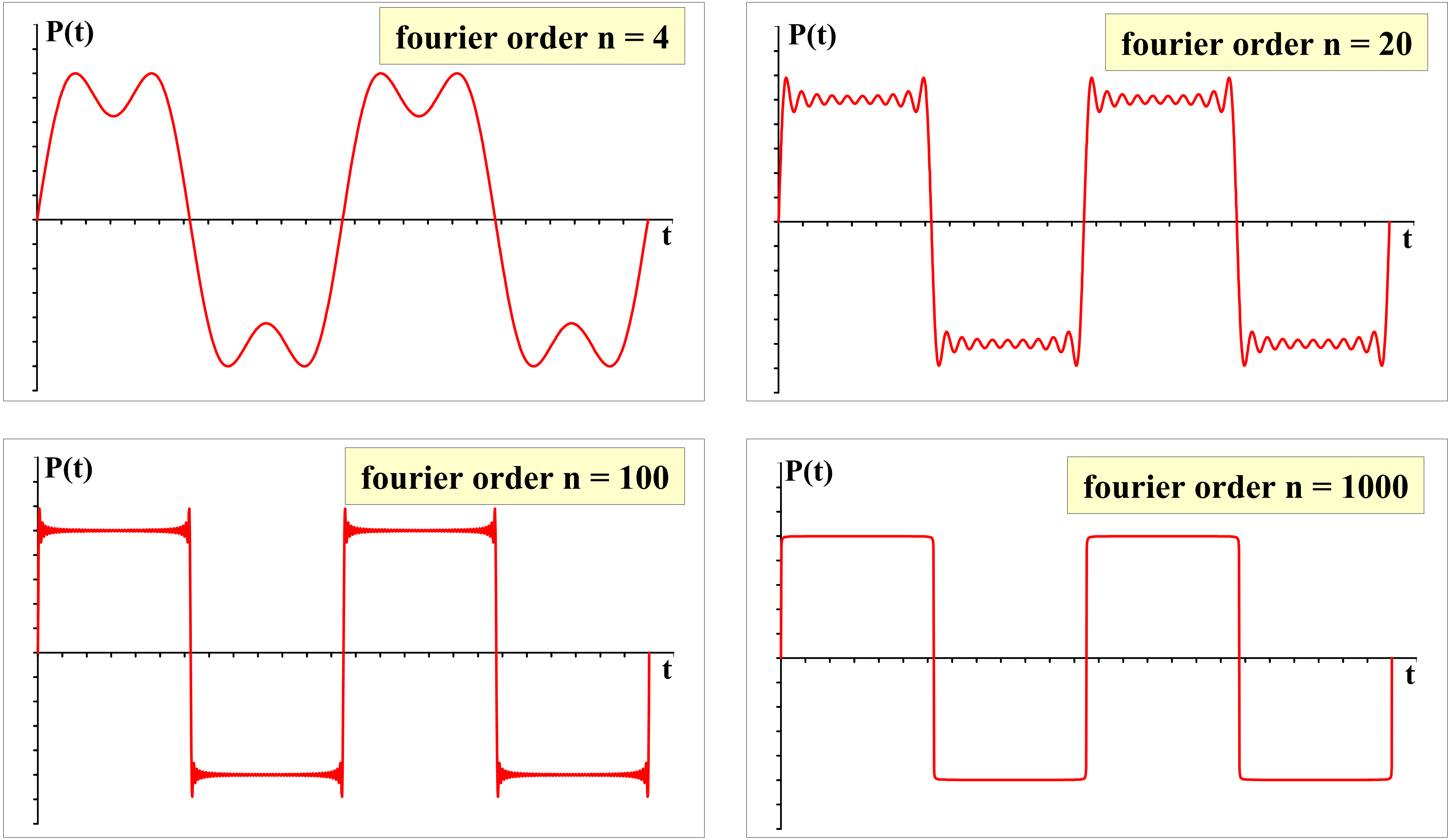


$$\Omega_P = \frac{2\pi}{T_P}$$

$$b_n = \frac{4}{T_P}\int_0^{T_P/2} P_0 \sin(n\Omega_P t)\,dt = P_0 \frac{4}{T_P}\frac{T_P}{n\,2\pi}\left[-\cos(n\Omega_P t)\right]_0^{T_P/2} = P_0 \frac{2}{n\,\pi}(1-\cos n\pi)$$

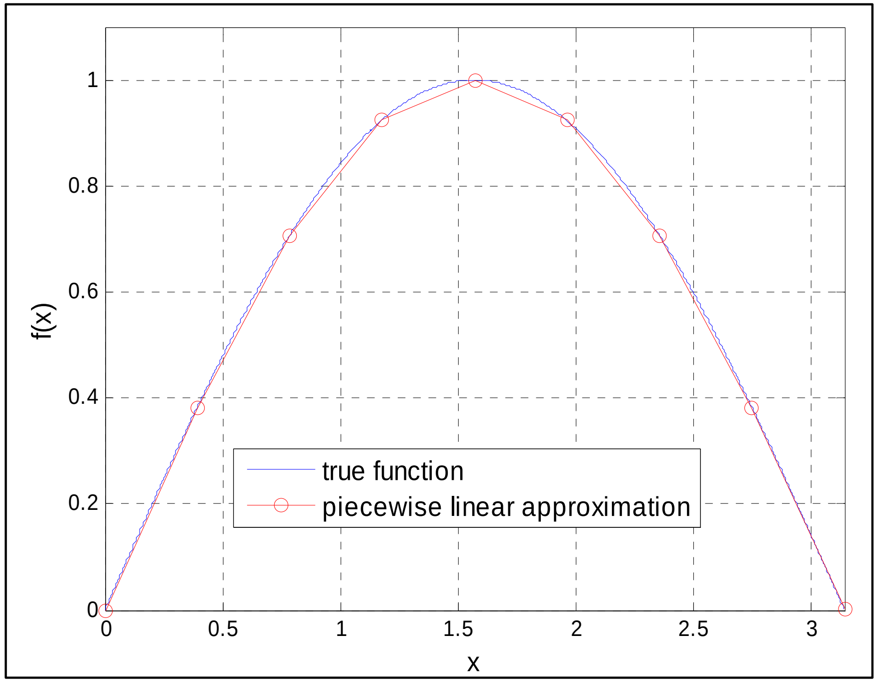
$$\Rightarrow \quad P(t) = \frac{4P_0}{\pi}\left\{\sin\Omega_P t + \frac{1}{3}\sin 3\Omega_P t + \frac{1}{5}\sin 5\Omega_P t + \frac{1}{7}\sin 7\Omega_P t + \ldots\right\}$$

Rectangular Sawtooth: Numerical Study



Example II: Discrete Data

The loading is often not given by analytical functions, but by *discrete data from measurements*. Then it is not possible to derive a closed solution for the Fourier coefficients. The Fourier integrals must instead be computed numerically by substituting the integrals by finite sums, e.g. the well-known trapezoidal rule. Here the true function is approximated in a piece-wise linear manner and the surface under the function is then the sum of the surfaces of the individual trapezes.

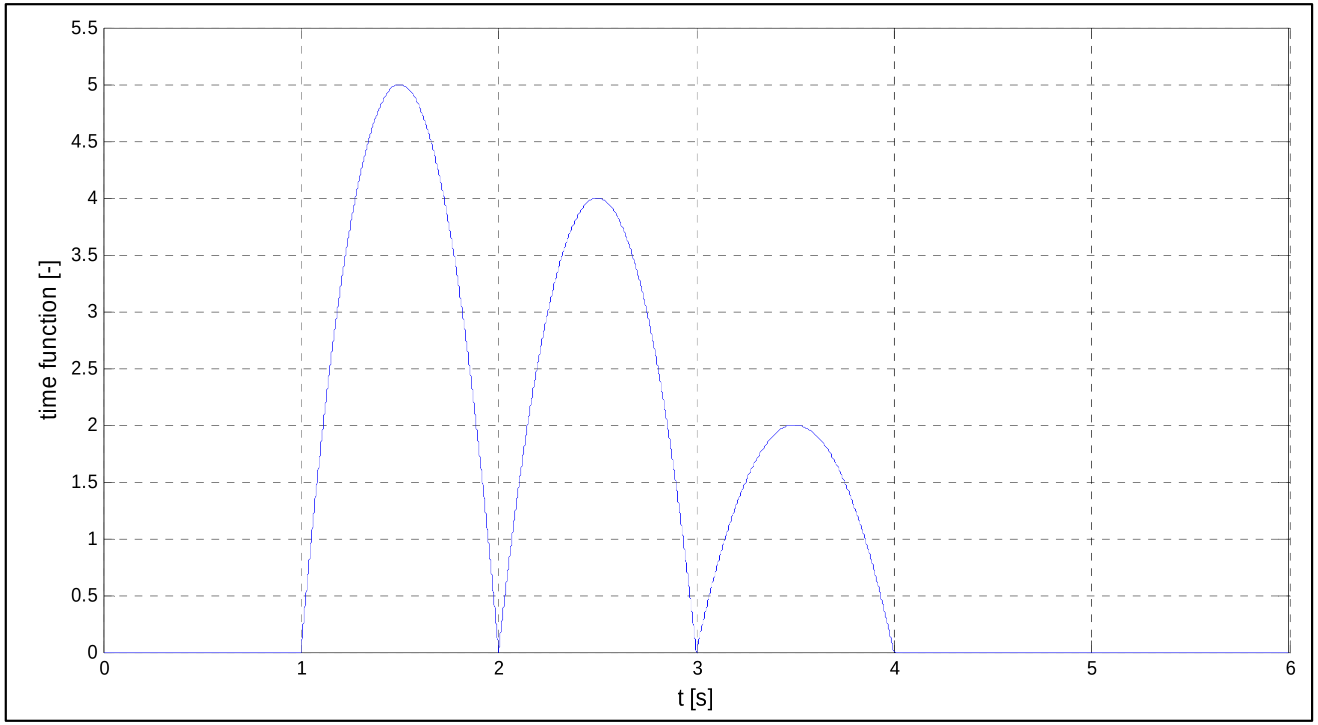


$$\int_{x=a}^{x=b} f(x)dx = 0.5\Delta x\left(f_1 + 2f_2 + 2f_3 + 2f_4 + \ldots + 2f_{n-1} + f_n\right)$$

The integration error can be reduced to an arbitrarily low value by increasing the number of integration intervals

Discrete Data: Series of Parabolas

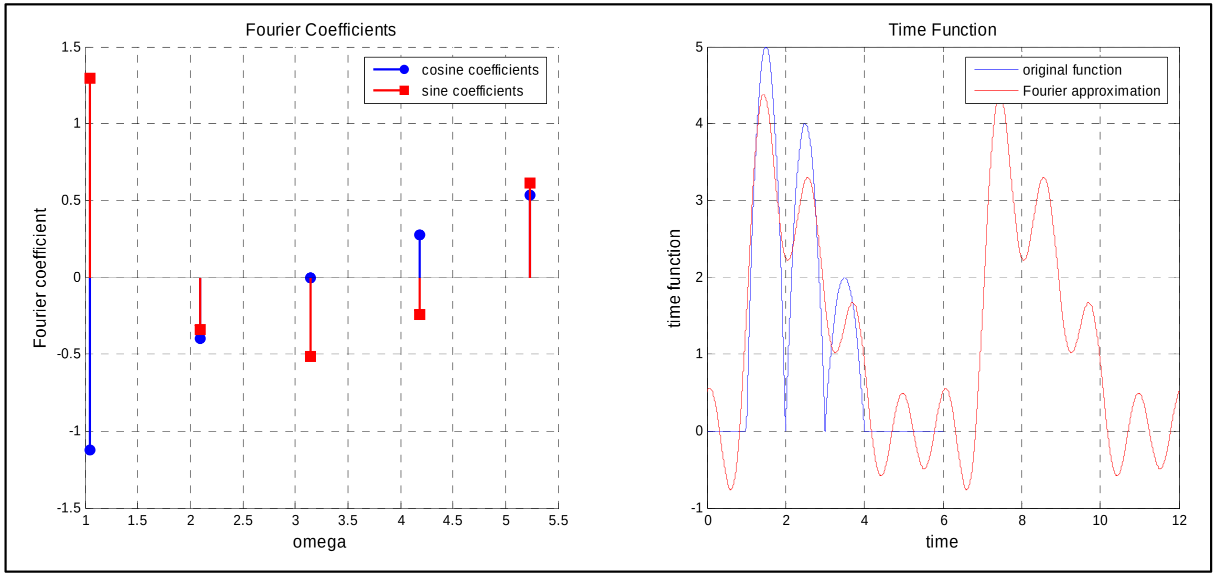
Our discrete load data take the form of a series of three parabolas of different amplitudes, pre- and anteceeded by zero loading. This load pattern is supposed to repeat itself after 6 seconds. Instead of solving the Fourier integrals analytically which would be possible here, we subdivide the entire period of $T_P = 6$ s into 6000 time intervals of $\Delta t = 0.001$ s each and compute the Fourier coefficients by the trapeziodal rule.



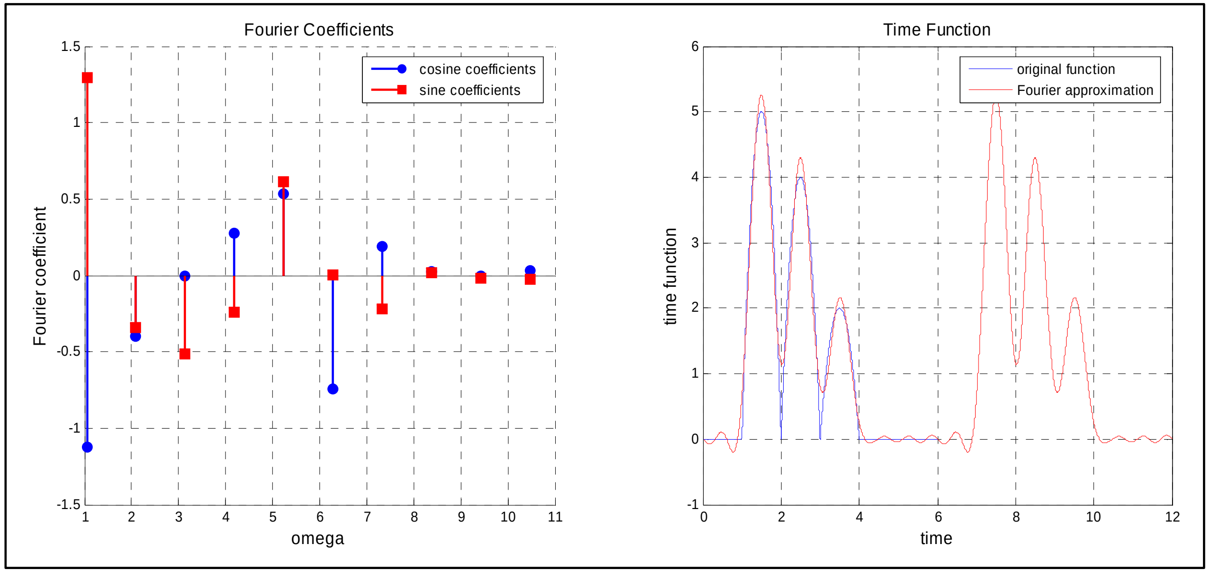
The constant term a_0 is independent of the frequency and is computed to:

$$a_0 = 2.4444$$

5 and 10 Fourier Terms

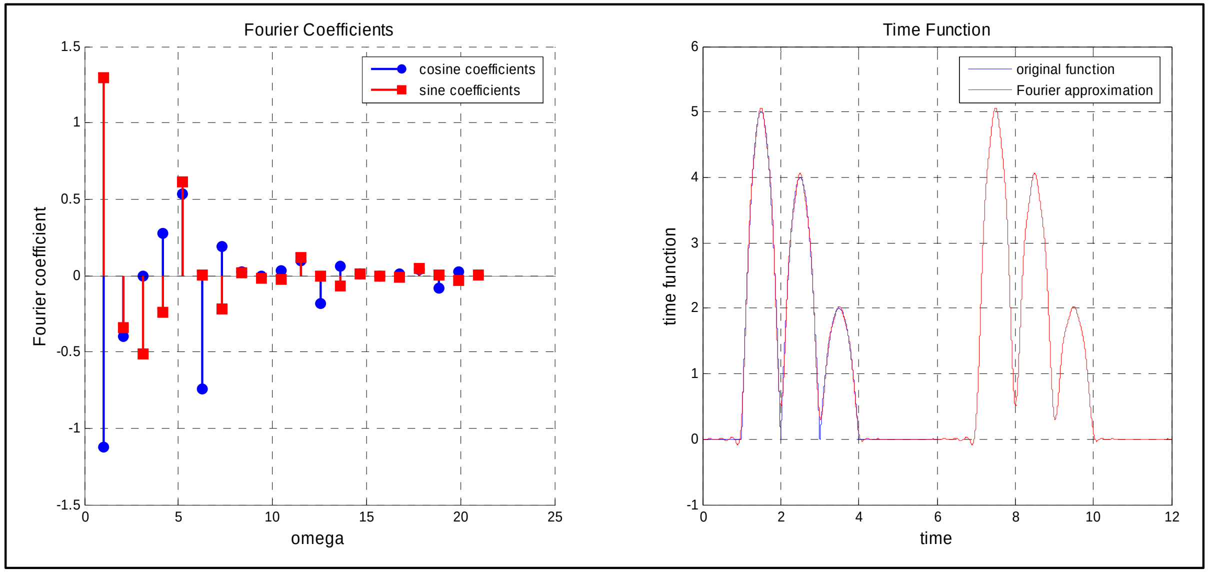


5 Fourier terms

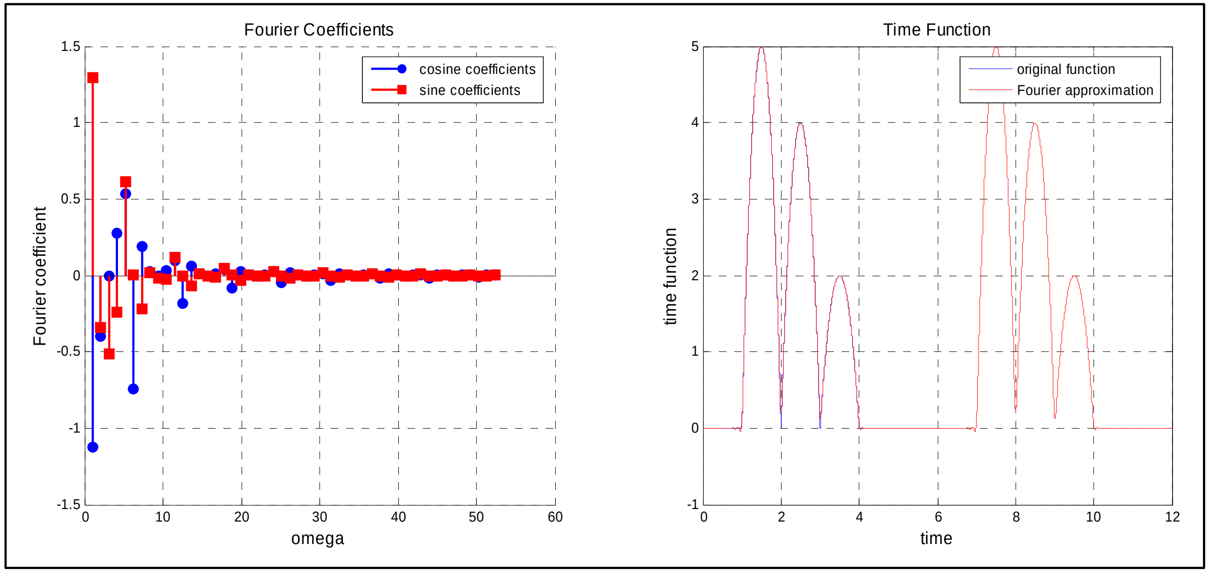


10 Fourier terms

20 and 50 Fourier Terms

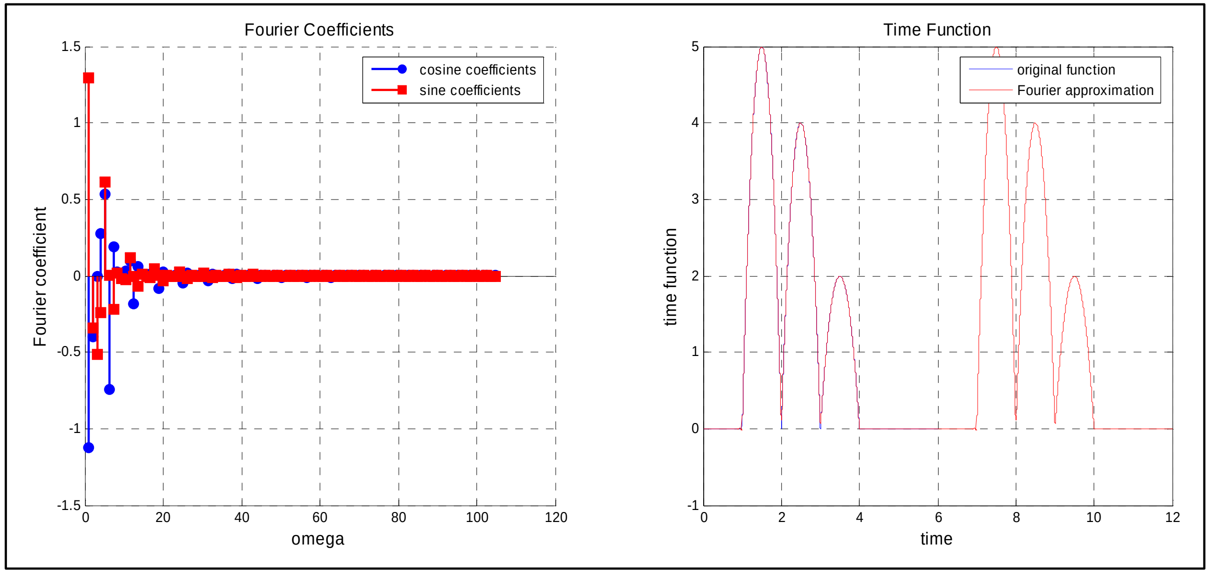


20 Fourier terms

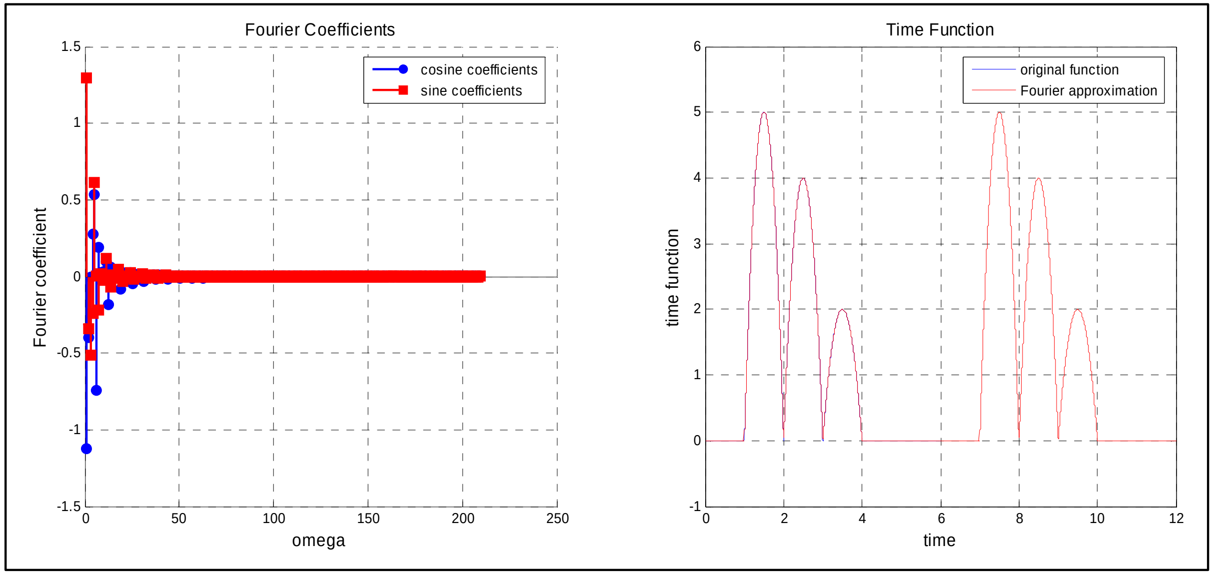


50 Fourier terms

100 and 200 Fourier Terms



100 Fourier terms



500 Fourier terms

Fourier Series: Amplitude/Phase Format

sine/cosine format

$$P(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \Omega_n t + \sum_{n=1}^{\infty} b_n \sin \Omega_n t$$

amplitude/phase format

$$P(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \sin(\Omega_n t + \Psi_n)$$

amplitudes: $c_n = \sqrt{a_n^2 + b_n^2}$

phases: $\tan \Psi_n = \frac{a_n}{b_n}$

Time Domain – Frequency Domain

function P

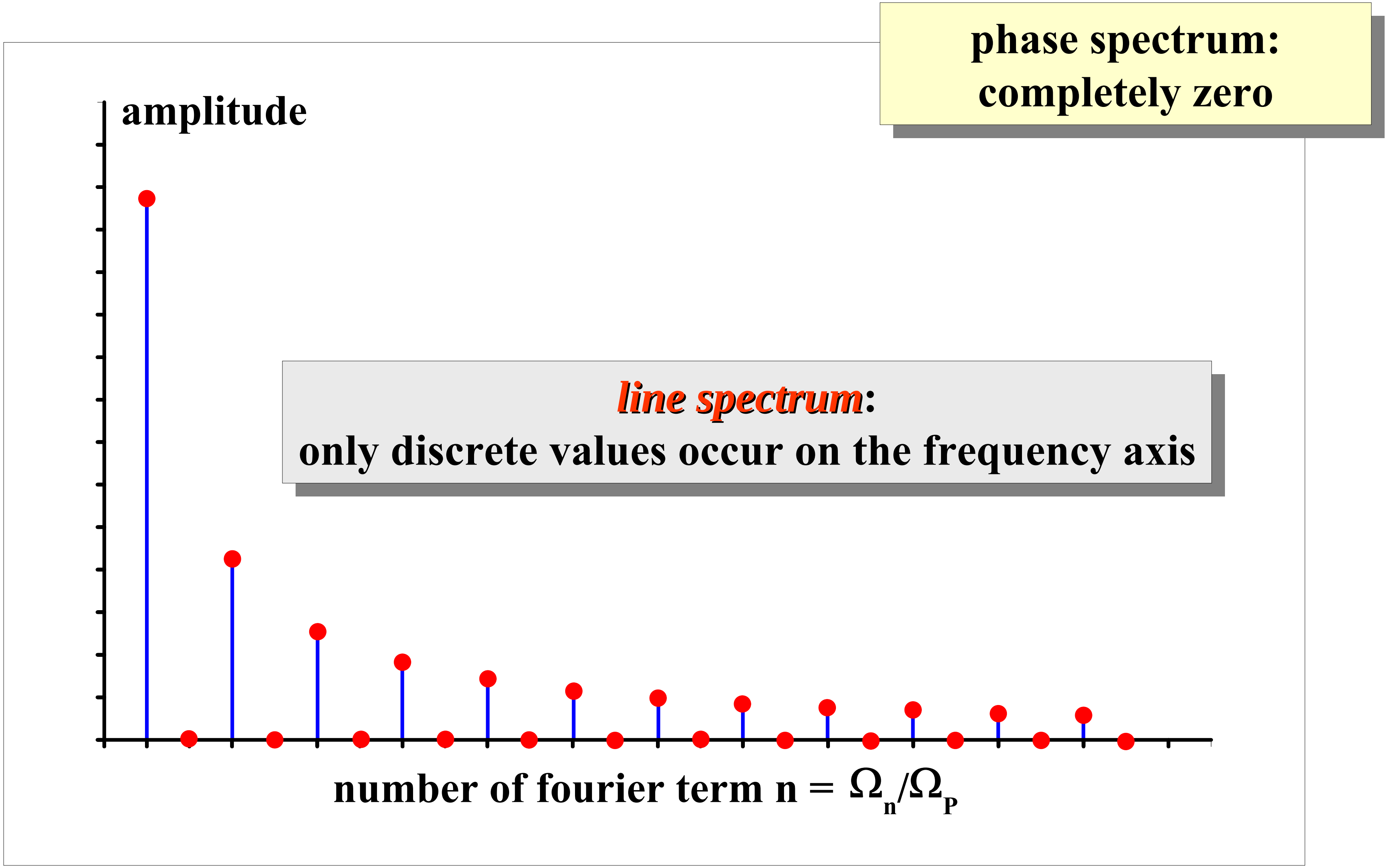
time domain representation:	frequency domain representation:
$P(t)$	$P(c_n, \psi_n)$

$c_n(\Omega)$: amplitude spectrum
$\psi_n(\Omega)$: phase spectrum

Any given periodic function is completely defined either by its

- **amplitude spectrum and phase spectrum in the frequency domain or its**
- **values in the time domain.**

Example: Sawtooth



Fourier Series: Complex Format

complex format

$$P(t) = \sum_{-\infty}^{\infty} \underline{c}_n \, e^{i\Omega_n t}$$

complex FOURIER coefficients

$$\underline{c}_n = \frac{1}{T} \int_{-T/2}^{T/2} P(t)\, e^{-i\Omega_n t} dt$$

relationship real/complex formats:

$$\underline{c}_n = \begin{cases} 0.5(a_n - ib_n) & \text{for } n \geq 0 \\ 0.5(a_n + ib_n) & \text{for } n < 0 \end{cases}$$

$$a_n = 2\,\mathrm{Re}(\underline{c}_n)$$
$$b_n = -2\,\mathrm{Im}(\underline{c}_n)$$

Complex Format: Proof

$$P(t)=\sum_{n=-\infty}^{\infty}\underline{C}_n e^{i\Omega_n t}=0.5(a_0-ib_0)+\sum_{n=-\infty}^{-1}0.5(a_n+ib_n)(\cos\Omega_n t+i\sin\Omega_n t)+\sum_{n=1}^{\infty}0.5(a_n-ib_n)(\cos\Omega_n t+i\sin\Omega_n t)$$

$$P(t)=0.5(a_0-ib_0)+0.5\sum_{n=-\infty}^{-1}(a_n\cos\Omega_n t+ia_n\sin\Omega_n t+ib_n\cos\Omega_n t-b_n\sin\Omega_n t)$$
$$+0.5\sum_{n=1}^{\infty}(a_n\cos\Omega_n t+ia_n\sin\Omega_n t-ib_n\cos\Omega_n t+b_n\sin\Omega_n t)$$

$$\sin(-x)=-\sin(x),\ \ \cos(-x)=\cos(x),\ \ b_0=0$$

$$P(t)=0.5a_0+\sum_{n=1}^{\infty}(a_n\cos\Omega_n t+b_n\sin\Omega_n t)$$

Summary & Outlook

We have seen that any periodic function can be approximated by an infinite series of harmonic parts: by a ***Fourier series***. The quality of the approximation depends on the number of terms we take into account.

This mathematical property can be interpreted for structural dynamics such that any ***periodic loading*** is made up of an infinite number of ***harmonics***. These act simultaneously, similar to load cases. The total response then is the sum of the partial responses induced by the individual harmonics.

The Fourier series is restricted to periodic functions, so ***transient loading***, e.g. the impact of a ship hitting a bridge pylon, cannot be captured. It is, however, possible to extend the Fourier concept to ***non-periodic functions***: then we have the ***Fourier transform***. The Fourier transform will be treated in Part 2 of this mathematical review.