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Lecture Series:

Structural Dynamics

Mathematical Review 1:

Complex Numbers

I: Algebraic Formulation

definition: imaginary number i:

$$i = \sqrt{-1}$$

complex number:

$$\underline{z} = a + ib$$

a: real part Re($\underline{z}$)
b: imaginary part Im($\underline{z}$)

conjugate complex number:

$$\tilde{\underline{z}} = a - ib$$

Algebraic Operations

given: 2 complex numbers

$\underline{z}_1 = a + ib$

$\underline{z}_2 = c + id$

addition:

$$\underline{z}_1 + \underline{z}_2 = (a+c) + i(b+d)$$

multiplication:

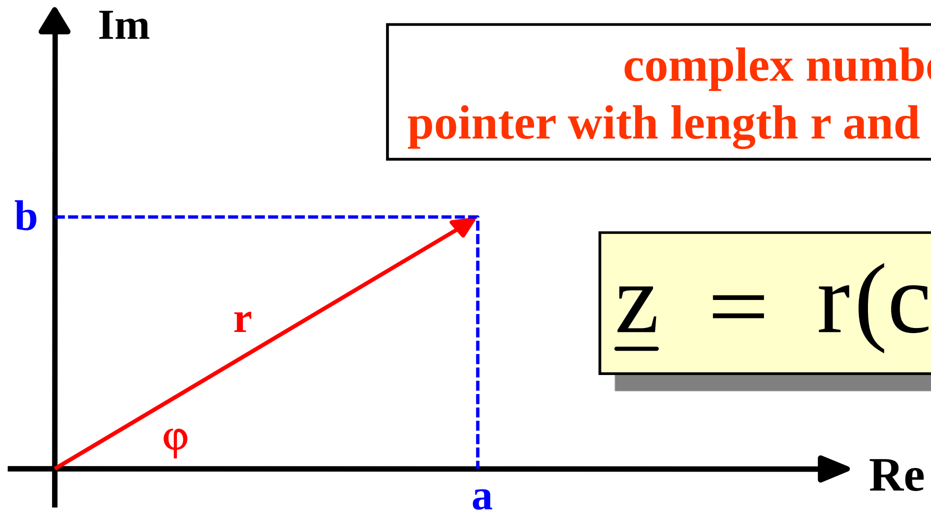
$$\underline{z}_1 \cdot \underline{z}_2 = (ac - bd) + i(ad + bc)$$

division:

$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{a+ib}{c+id} \cdot \frac{c-id}{c-id} = \frac{(ac+bd) + i(bc-ad)}{c^2 + d^2}$$

II: Polar Formulation

complex number plane by GAUSS



complex number:
pointer with length r and orientation φ

$$\underline{z} = r(\cos\varphi + i\sin\varphi)$$

absolute value r of $\underline{z}$: $r = |\underline{z}| = \sqrt{a^2 + b^2}$

phase φ of $\underline{z}$: $\tan\varphi = b/a$

$\Rightarrow$

$a = r\cos\varphi$

$b = r\sin\varphi$

Algebraic Operations

multiplication:

$$\underline{z}_1 \cdot \underline{z}_2 = r_1\, r_2\, (\cos\varphi_1 + i\sin\varphi_1)\,(\cos\varphi_2 + i\sin\varphi_2)$$

$$\underline{z}_1 \cdot \underline{z}_2 = r_1 r_2 \{(\cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2) + i(\sin\varphi_1 \cos\varphi_2 + \cos\varphi_1 \sin\varphi_2)\}$$

$$\underline{z}_1 \cdot \underline{z}_2 = r_1\, r_2\, (\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$$

Multiplication becomes very easy in the GAUSSIAN number plane: the magnitude of the product is equal to the product of the magnitudes, and the angle of the product is equal to the sum of the angles.

Algebraic Operations

division:

$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)}$$

$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{r_1}{r_2}\frac{(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 - i\sin\varphi_2)}{(\cos\varphi_2 + i\sin\varphi_2)(\cos\varphi_2 - i\sin\varphi_2)}$$

$$\frac{\underline{z}_1}{\underline{z}_2} == \frac{r_1}{r_2}\frac{\{(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2) + i(\cos\varphi_2\sin\varphi_1 - \cos\varphi_1\sin\varphi_2)\}}{\cos^2\varphi_2 + \sin^2\varphi_2)}$$

$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$$

III: Exponential Formulation

EULER's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Proof:

$$y_1 = e^{i\varphi}$$

$$y_2 = \cos\varphi + i\sin\varphi$$

$$y_1' = \frac{dy_1}{d\varphi} = ie^{i\varphi}$$

$$y_2' = \frac{dy_2}{d\varphi} = -\sin\varphi + i\cos\varphi = i(\cos\varphi + i\sin\varphi)$$

$$y_1' = iy_1$$

$$y_2' = iy_2$$

Both y_1 and y_2 are solutions of the same differential equation. Therefore they must be identical and thus the EULER identity has been proven to be true.

Algebraic Operations

conjugate complex number:

$$\underline{z} = r\,e^{i\varphi} \;\Rightarrow\; \underline{\tilde{z}} = r\,e^{-i\varphi}$$

multiplication and division:

$$\underline{z}_1 \cdot \underline{z}_2 = r_1\,r_2\,e^{i(\varphi_1 + \varphi_2)}$$

$$\frac{\underline{z}_1}{\underline{z}_2} = \frac{r_1}{r_2}\,e^{i(\varphi_1 - \varphi_2)}$$

Powers and Roots

n-th power of a complex number:

$$\underline{z} = re^{i\varphi} \Rightarrow \underline{z}^n = (re^{i\varphi})^n = r^n e^{i\varphi n}$$

A complex number is raised to the n-th power by raising the magnitude to the n-th power and multiplying the angle by n. Very easy in the polar and complex form.
Computing the n-th root is achieved by extracting the n-th root of the magnitude and dividing the angle by n.

roots of complex numbers:

$$\underline{z} = re^{i\varphi} \Rightarrow \sqrt[n]{\underline{z}} = (re^{i\varphi})^{\frac{1}{n}} = \sqrt[n]{r}e^{\frac{i\varphi}{n}}$$