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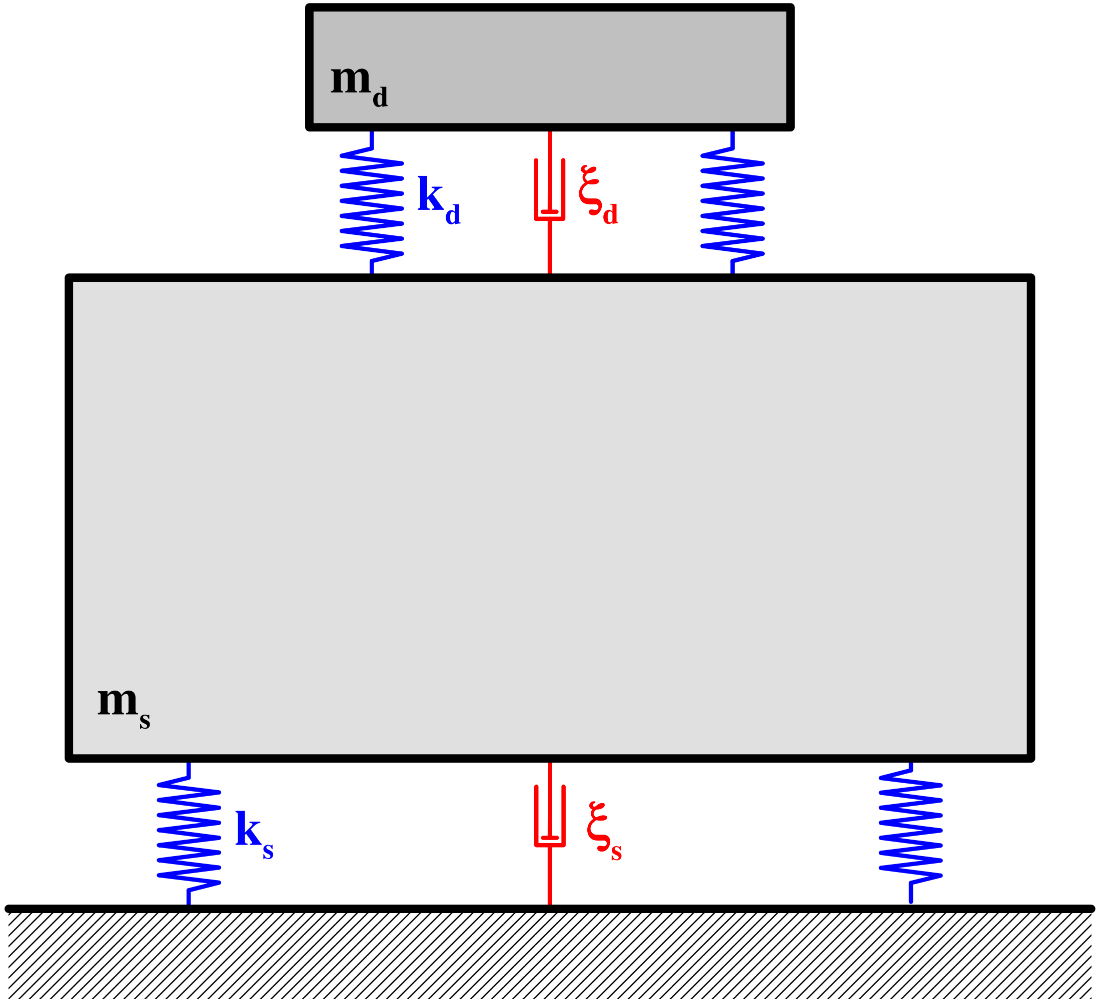
Tuned Mass Dampers in the Design of Light-Weight Structures

Theory, Numerical Simulation, Application

Lecture 02
Optimization Techniques for TMDs

Structural Mechanics and Numerical Methods
Bergische Universität Wuppertal

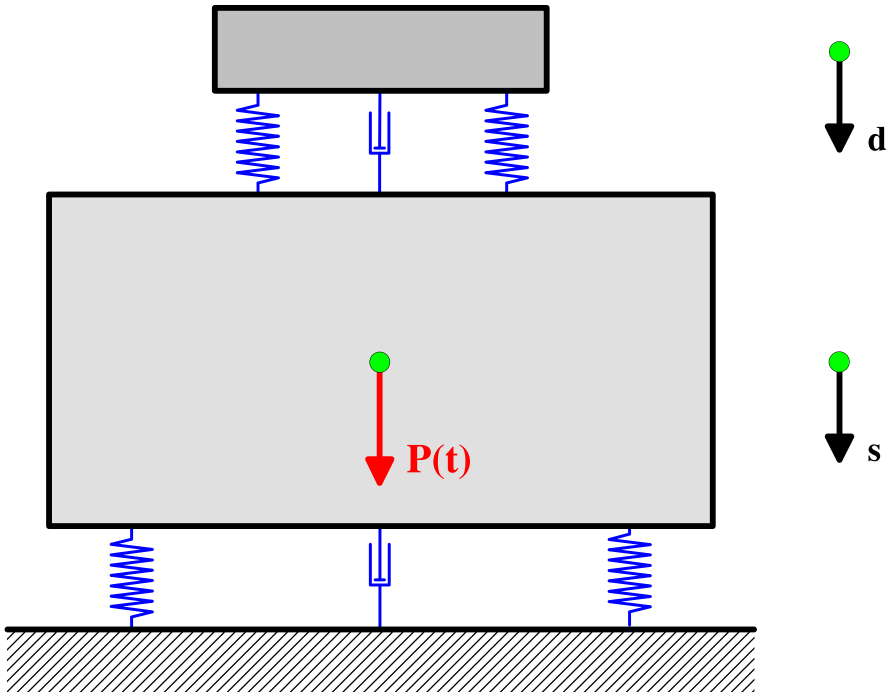
Idealization as a 2DOF-System



tuned mass damper:
mass m_d
stiffness k_d
damping ξ_d

structural model:
mass m_s
stiffness k_s
damping ξ_s

Coupled Equations of Motion - V1



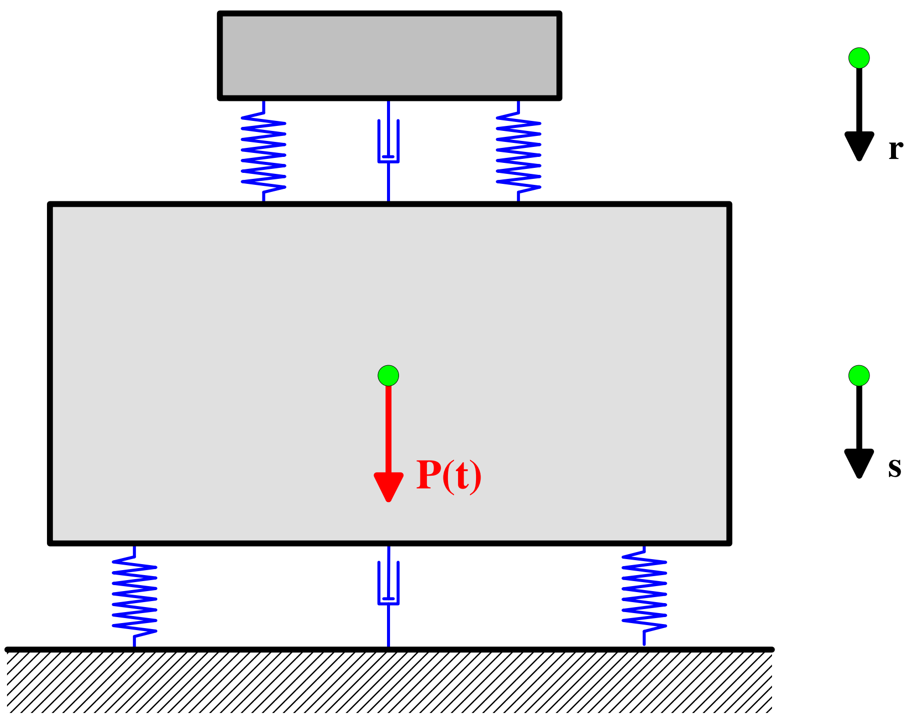
dof s:
absolute displacement of the system

dof d:
absolute displacement of the damper

system of differential equations

$$\begin{bmatrix} m_s & 0 \\ 0 & m_d \end{bmatrix} \begin{bmatrix} \ddot{s} \\ \ddot{d} \end{bmatrix} + \begin{bmatrix} c_s + c_d & -c_d \\ -c_d & c_d \end{bmatrix} \begin{bmatrix} \dot{s} \\ \dot{d} \end{bmatrix} + \begin{bmatrix} k_s + k_d & -k_d \\ -k_d & k_d \end{bmatrix} \begin{bmatrix} s \\ d \end{bmatrix} = \begin{bmatrix} P(t) \\ 0 \end{bmatrix}$$

Coupled Equations of Motion – V2



dof s:
absolute displacement of the system

dof r:
relative displacement of the damper

system of differential equations

$$\begin{bmatrix} m_s & 0 \\ m_d & m_d \end{bmatrix}\begin{bmatrix} \ddot{s} \\ \ddot{r} \end{bmatrix}+\begin{bmatrix} c_s & -c_d \\ 0 & c_d \end{bmatrix}\begin{bmatrix} \dot{s} \\ \dot{r} \end{bmatrix}+\begin{bmatrix} k_s & -k_d \\ 0 & k_d \end{bmatrix}\begin{bmatrix} s \\ r \end{bmatrix}=\begin{bmatrix} P(t) \\ 0 \end{bmatrix}$$

Design Variables of the TMD

TMD is characterised by: m_d, k_d, c_d

mass ratio μ:

$$\mu = \frac{m_d}{m_s}$$

tuning ratio κ:

$$\kappa = \frac{\omega_d}{\omega_s}$$

damping ratio ξ:

$$\xi_d = \frac{c_d}{2\omega_d m_d} = \frac{c_d}{c_{d,crit}}$$

eigenfrequencies:

$$\omega_d = \sqrt{\frac{k_d}{m_d}}$$

$$\omega_s = \sqrt{\frac{k_s}{m_s}}$$

Problem:

How do we choose the design parameters for an optimum performance of the TMD? What is an optimum performance?

Why and how Does a TMD Work?

In the following we will derive, mathematically, equations which allow us to design TMDs with desired properties. These equations and their graphic representations demonstrate the influence of certain parameters on the performance of TMDs, yet they do not answer really help us to understand on a non-mathematical level why a TMD works at all. So before we start the math, we will study a simple example: *harmonic resonant* excitation of the structure

$$V_{1,res} = \frac{1}{2\xi}$$

For resonance we have the maximum dynamic amplification which goes, if there is no damping, to infinity. So the question is: how does the addition of a very small mass change the dynamic behavior of the coupled system? We can solve the equation of motion in the time domain analytically and see what happens.

Coupled System: Example

We study a structure with unit mass and an eigenfrequency of f_s=1.0 Hz. The damper shall have a very small mass (μ=0.0001). Its eigenfrequency shall be identical to that of the structure (κ=1). We will compute the response in modal space, so we need to use the variant V1 with the FE degrees of freedom since the modal superposition method assumed the symmetry of the system matrices. Damping is set to zero for both structure and TMD.

$$m_s = 1.0\,\text{ton} \;\Rightarrow\; k_s = (2\pi f_s)^2 \cdot m_s = 39.4784\,\text{kN/m}$$

$$\mu = 0.0001 \;\Rightarrow\; m_d = \mu \cdot m_s = 10^{-4}\,\text{tons}$$

$$\kappa = 1 \;\Rightarrow\; f_d = f_s \;\Rightarrow\; k_d = (2\pi f_s)^2 \cdot m_d = 39.4784 \cdot 10^{-4}\,\text{kN/m}$$

Eigenfrequency Analysis: Equations

matrices in V1-formulation

$$\mathbf{K} = \begin{bmatrix} k_s + k_d & -k_d \\ -k_d & k_d \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m_s & 0 \\ 0 & m_d \end{bmatrix}$$

eigenvalue problem in V1-formulation

$$\left\{ \begin{bmatrix} k_s + k_d & -k_d \\ -k_d & k_d \end{bmatrix} - \omega^2 \begin{bmatrix} m_s & 0 \\ 0 & m_d \end{bmatrix} \right\} \begin{bmatrix} \hat{s} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigenfrequency Analysis: Results

eigenmode 1:

$$f_1 = 0.995012\ \text{Hz}$$

$$\begin{bmatrix} \hat{s} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} -1.000000 \\ -100.501250 \end{bmatrix}$$

Structure and damper move in phase!

eigenmode 2:

$$f_2 = 1.005012\ \text{Hz}$$

$$\begin{bmatrix} \hat{s} \\ \hat{v} \end{bmatrix} = \begin{bmatrix} -1.000000 \\ 99.501250 \end{bmatrix}$$

Structure and damper move out of phase!

Transformation into Modal Space

modal mass:

$$\mathbf{M}_{\text{mod}} = \mathbf{\Phi}^{\text{T}} \cdot \mathbf{M} \cdot \mathbf{\Phi} = \begin{bmatrix} 2.0101 & 0 \\ 0 & 1.9900 \end{bmatrix}$$

modal stiffness:

$$\mathbf{K}_{\text{mod}} = \mathbf{\Phi}^{\text{T}} \cdot \mathbf{K} \cdot \mathbf{\Phi} = \begin{bmatrix} 78.5640 & 0 \\ 0 & 79.3536 \end{bmatrix}$$

modal loads:

$$\mathbf{P}_{\text{mod}} = \mathbf{\Phi}^{\text{T}} \cdot \mathbf{P} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Harmonic Excitation

We excite the structure with a harmonic load since we have a closed analytical solution. We are only interested in the stationary, long-time vibration. The load frequency *Ω* is chosen to coincide with the eigenfrequency of the structure without TMD.

$$P(t) = \hat{P} \cdot \sin(\Omega t)$$ ⇨ $$V(t) = V_1 \frac{\hat{P}}{k} \cdot \sin(\Omega t - \varphi)$$

dynamic amplification:

$$V_1 = \frac{1}{\sqrt{(2\xi\eta)^2 + (1-\eta^2)^2}}$$

phase angle:

$$\tan\varphi = \frac{2\xi\eta}{1-\eta^2}$$

Solution in Modal Space

parameters for the two modes:

$$\eta = \begin{bmatrix} F_{load}/f_1 \\ F_{load}/f_2 \end{bmatrix} = \begin{bmatrix} 1.005 \\ 0.995 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 99.5012 \\ 100.5012 \end{bmatrix}$$

$$\varphi = \begin{bmatrix} 3.1416 \\ 0.0000 \end{bmatrix}$$

$$V_{stat} = \begin{bmatrix} P_{mod1}/k_{mod1} \\ P_{mod2}/k_{mod2} \end{bmatrix} = \begin{bmatrix} -0.0127 \\ -0.0126 \end{bmatrix}$$

$$V_{dyn} = \begin{bmatrix} 1.2665 \\ 1.2665 \end{bmatrix}$$

mode 1:

$$V_{mod,1}(t) = 1.2665 \cdot \sin(\Omega t - 3.1416)$$

mode 2:

$$V_{mod,2}(t) = 1.2665 \cdot \sin(\Omega t)$$

mode 1:

$$\sin(\Omega t - \pi) = \sin(\Omega t)\cos(\pi) - \cos(\Omega t)\sin(\pi) = -\sin(\Omega t)$$

$$V_{mod,1}(t) = -1.2665 \cdot \sin(\Omega t)$$

Solution for the True System

synthesis of the modal solutions:

$$\begin{bmatrix} s(t) \\ d(t) \end{bmatrix} = V_{mod,1} \cdot \mathbf{\Phi}_1 + V_{mod,2} \cdot \mathbf{\Phi}_2$$

harmonic excitation:

$$\begin{bmatrix} s(t) \\ d(t) \end{bmatrix} = -1.2665 \sin(\Omega t) \cdot \begin{bmatrix} -1.000000 \\ -100.501250 \end{bmatrix} + 1.2665 \sin(\Omega t) \cdot \begin{bmatrix} -1.000000 \\ 99.501250 \end{bmatrix}$$

$$\begin{bmatrix} s(t) \\ d(t) \end{bmatrix} = 1.2665 \sin(\Omega t) \cdot \begin{bmatrix} 0 \\ 200.0025 \end{bmatrix}$$

Summary

We had two initially separate undamped systems – structure and damper – with exactly the same eigenfrequencies. Under a fully resonant harmonic load both would be excited to infinitely large amplitudes.

Now we connect them and get a coupled system with two degrees of freedom. The eigenfrequencies of the coupled system change only very little since the damper mass is extremely small. The infinitely large dynamic amplification is reduced to two large values since we no longer have perfect resonance. Both mode shapes oscillate with the same amplitudes, but perfectly out of shape.

As a result the vibration of the system is completely eliminated since the two mode shapes cancel out, while the TMD is excited to large oscillations! Nature chooses to excite the small mass and let the large mass remain at rest.

Eigenfrequencies

eigenvalue problem:

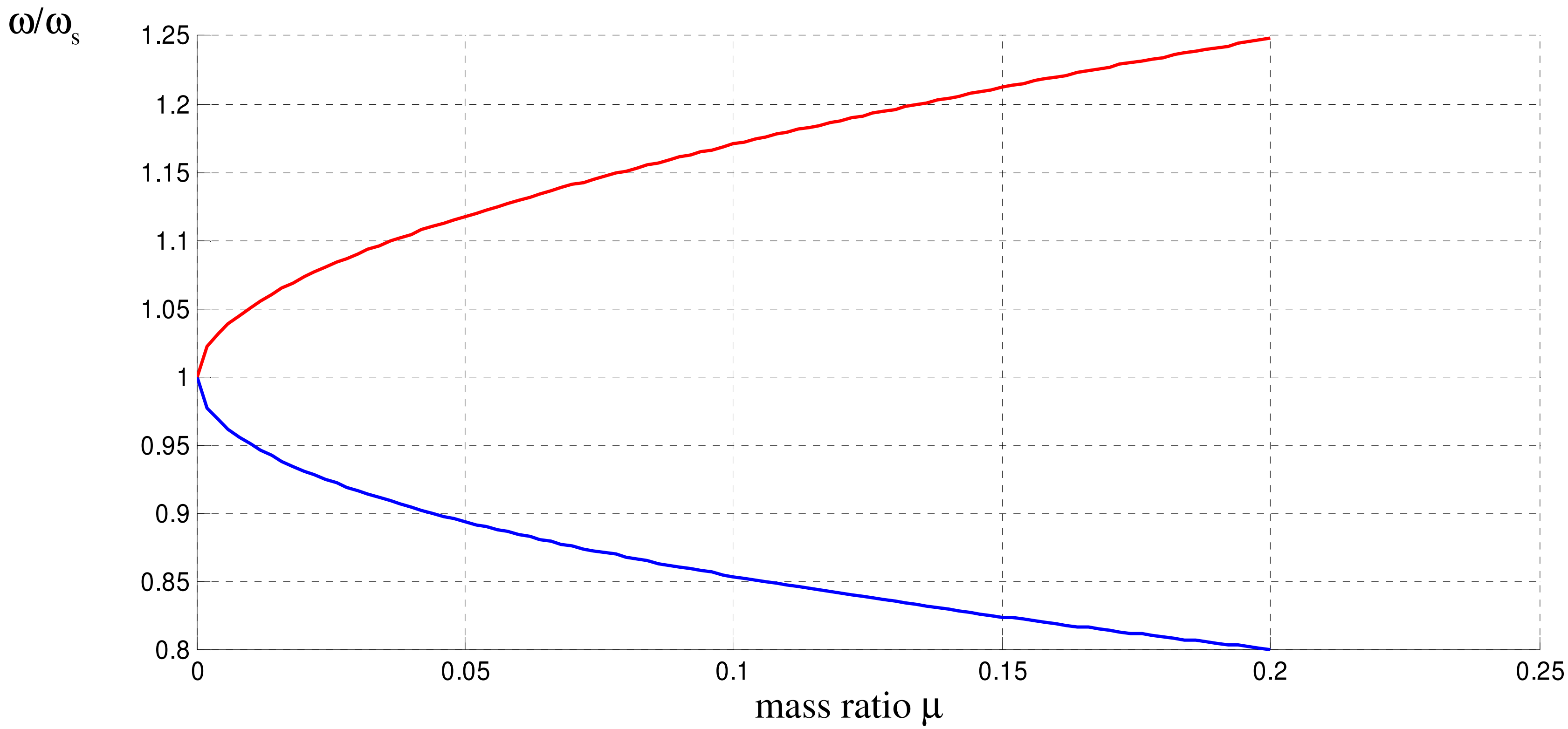
$$\left\{\begin{bmatrix}\omega_s^2 & -\mu\omega_d^2 \\ 0 & \omega_d^2\end{bmatrix} - \omega^2\begin{bmatrix}1 & 0 \\ 1 & 1\end{bmatrix}\right\}\begin{bmatrix}s \\ r\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

$$\frac{\omega_{1,2}}{\omega_s} = \sqrt{\frac{1}{2}\left\{\left(1+\kappa^2+\mu\kappa^2\right) \mp \sqrt{\left(1+\kappa^2+\mu\kappa^2\right)^2 - 4\kappa^2}\right\}}$$

The original single eigenfrequency of the system without TMD is split into two distict eigenfrequencies.

Eigenfrequencies as Functions of μ

frequency tuning $\kappa = 1$



The split of the eigenfrequencies grows more pronounced with increasing mass of the TMD. How does the coupled structure vibrate?

Optimization

A TMD can only be optimized with respects to:

- a specific response:
  - displacement
  - velocity
  - acceleration
- a specific load type:
  - harmonic load,
  - periodic load, ...

Solution for Harmonic Loads

harmonic load in complex formulation: $P(t) = \underline{\hat{P}} e^{i\Omega t}$

stationary response also harmonic: $s(t) = \underline{\hat{s}} e^{i\Omega t}, \; r(t) = \underline{\hat{r}} e^{i\Omega t}$

complex system of equations for the complex amplitudes:

$$\begin{bmatrix} \omega_s^2 - \Omega^2 + 2\xi_s\omega_s\Omega i & -\mu\omega_d^2 - 2\mu\xi_d\omega_d\Omega i \\ -\Omega^2 & \omega_d^2 - \Omega^2 + 2\xi_d\omega_d\Omega i \end{bmatrix} \begin{bmatrix} \underline{\hat{s}} \\ \underline{\hat{r}} \end{bmatrix} = \frac{\underline{\hat{P}}}{m_s} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Response for Harmonic Loads

auxiliary variables:

$$
\begin{aligned}
b_1 &= \kappa^2 - \eta^2 \\
b_2 &= 2\xi_d \eta \kappa \\
b_3 &= \eta^4 - \eta^2\left(1 + \kappa^2 + \mu\kappa^2 + 4\xi_s\xi_d\kappa\right) + \kappa^2 \\
b_4 &= \eta\left\{2\xi_s\left(\kappa^2 - \eta^2\right) + 2\xi_d\kappa\left(1 - \eta^2 - \mu\eta^2\right)\right\}
\end{aligned}
$$

static deformation:

$$s_{st} = \frac{\hat{P}}{k_s}$$

frequency ratio:

$$\eta = \frac{\Omega}{\omega_s}$$

dynamic amplification factors:

structure

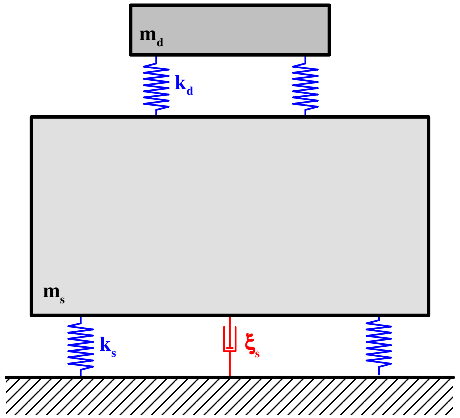
$$V_{1s} = \frac{s}{s_{st}} = \sqrt{\frac{b_1^2 + b_2^2}{b_3^2 + b_4^2}}$$

TMD

$$V_{1r} = \frac{r}{r_{st}} = \sqrt{\frac{\eta^4}{b_3^2 + b_4^2}}$$

Behaviour of an Undamped TMD

simplification: ξ_d = 0!



$$
\begin{aligned}
b_1 &= \kappa^2 - \eta^2 \\
b_2 &= 0 \\
b_3 &= \eta^4 - \eta^2\left(1 + \kappa^2 + \mu\kappa^2\right) + \kappa^2 \\
b_4 &= 2\xi_s\eta\left(\kappa^2 - \eta^2\right)
\end{aligned}
$$

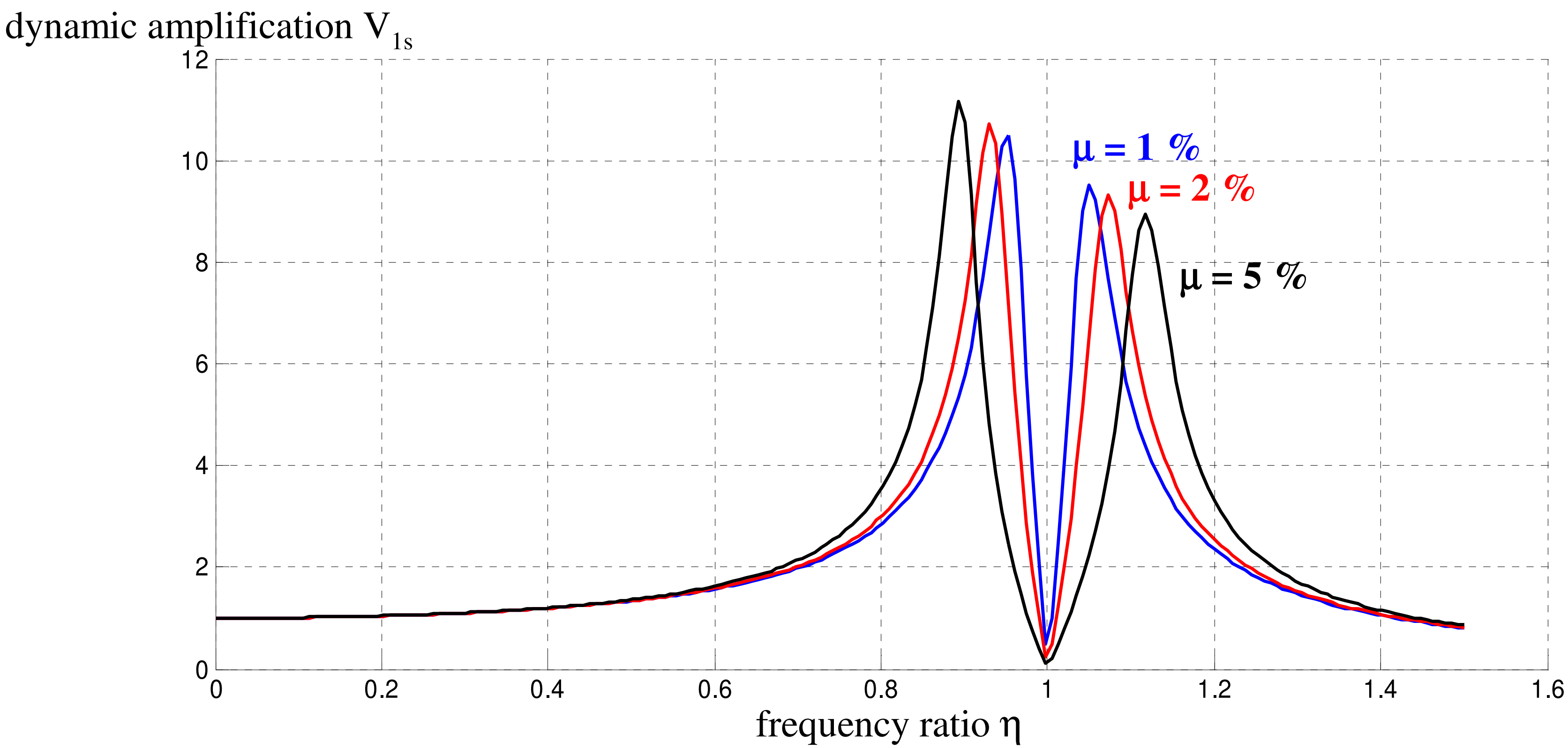
$$V_{1s} = \sqrt{\frac{b_1^2}{b_3^2 + b_4^2}}$$

The oscillation of the structure becomes zero if the damper has the same eigenfrequency as the load: ω_d = Ω.

The damping out of the motion has nothing to do with damping in the sense of energy dissipation. The TMD has no damping at all!

Response of the Structure

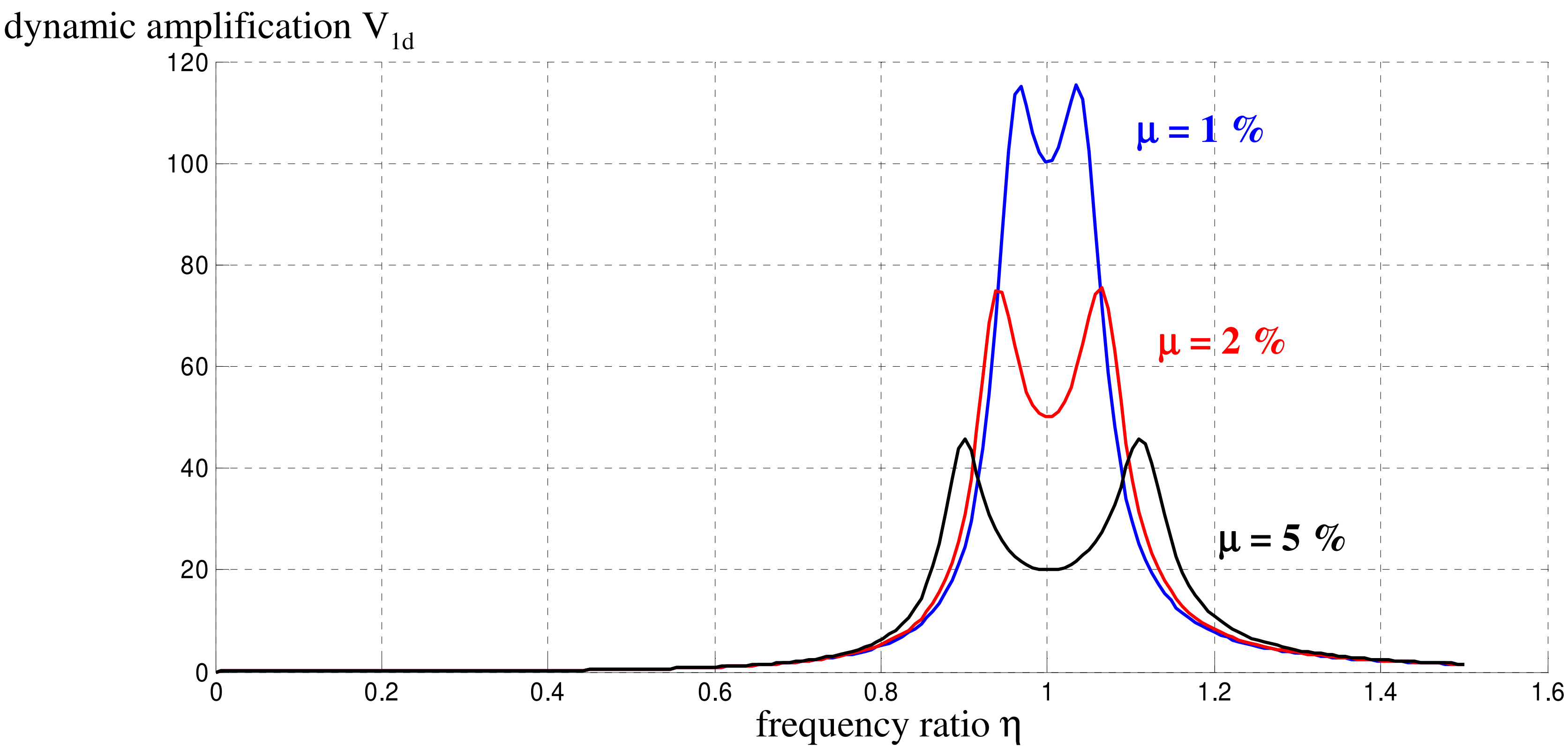
fixed parameters: κ = 1.0 and ξ_s = 0.05



Independently of μ is the oscillation with the load frequency completely damped out.

Response of the TMD

fixed parameters: κ = 1.0 and ξ_s = 0.05



The mass ratio μ influences the response of the TMD. A small TMD must produce large amplitudes in order to work.

What is an Optimum Performance?

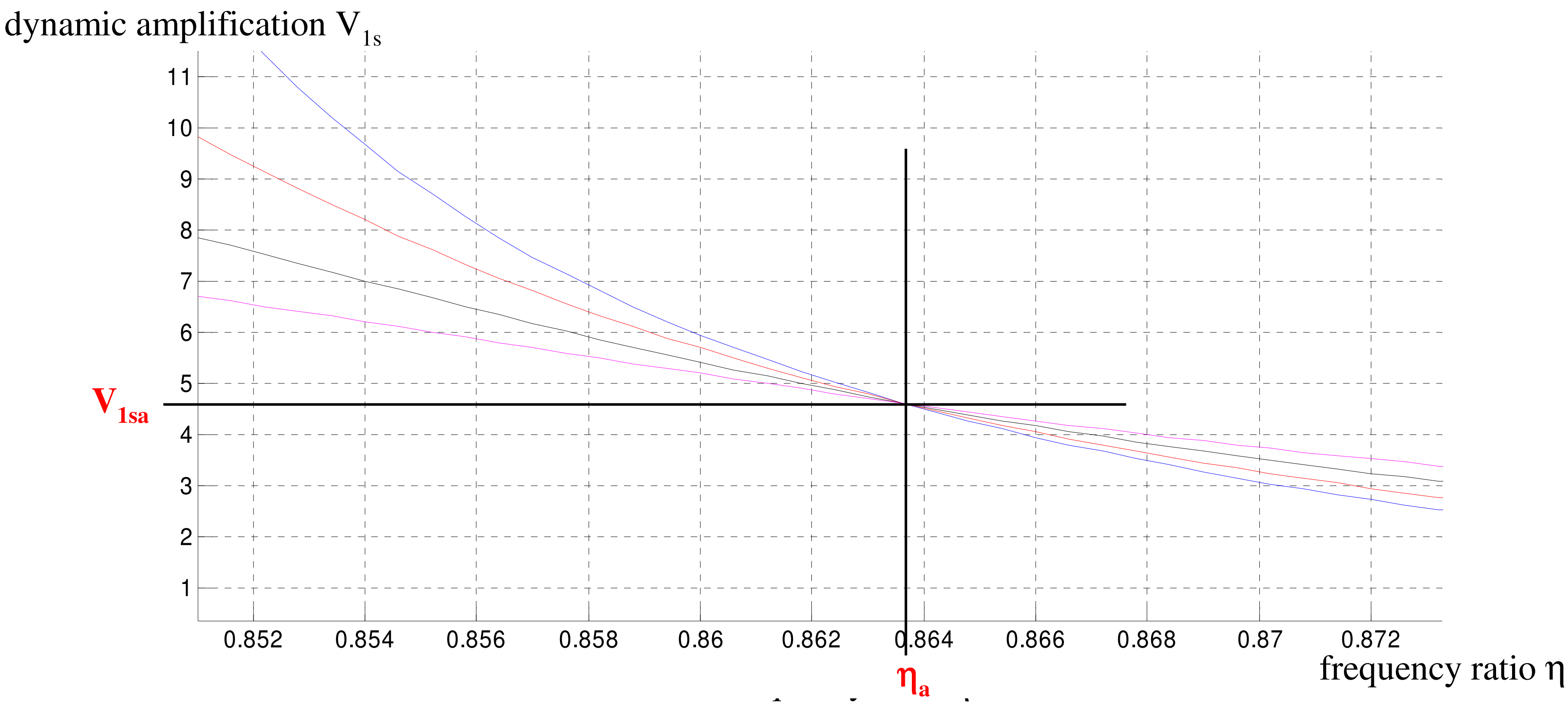
Naive design philosophy: The load frequency Ω is given; then the damper is designed as having the load as eigenfrequency and we have absolutely no vibration in the structure. The mass of the TMD is chosen such that its amplitudes are limited to reasonable values.

This design is not good!! If the load frequency were to shift only a little bit or the TMD is not adjusted perfectly, then the effectiveness of the TMD would fall off drastically.

Good design strategy: A compromise must be struck between effectivness and safety. The TMD must perform in a broad(er) band of frequencies with the same effectiveness to minimize the system sensitivity.

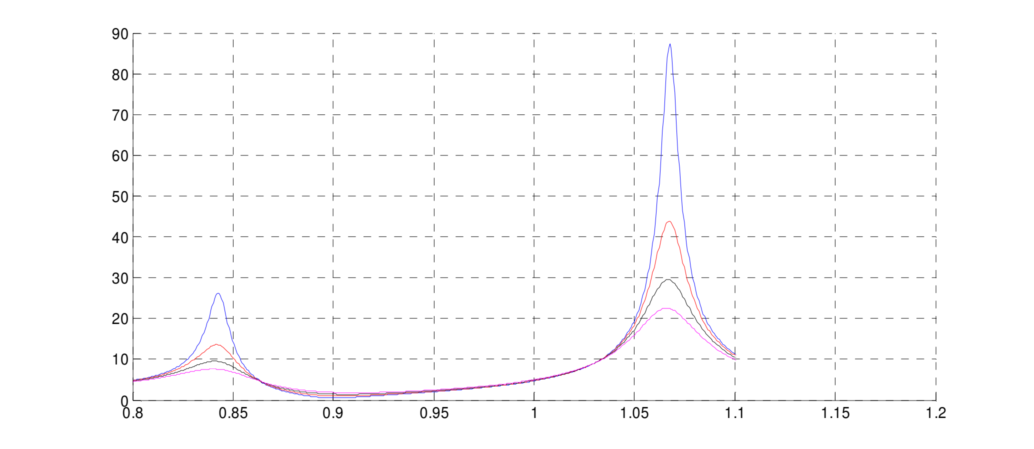
Observation 1: Common Points

fixed parameters: κ = 1.0, μ = 0.05, ξ_s = 0.05



All resonance curves pass through 2 common points η_a and η_b. The frequencies η_a and η_b as well as V_{1sa} and V_{1sb} depend on κ and μ.

Requirement 1: Symmetric Response



The sensitivity is not symmetric: the amplification is more sensitive with respect to positive rather than negative deviations from η_{target}.

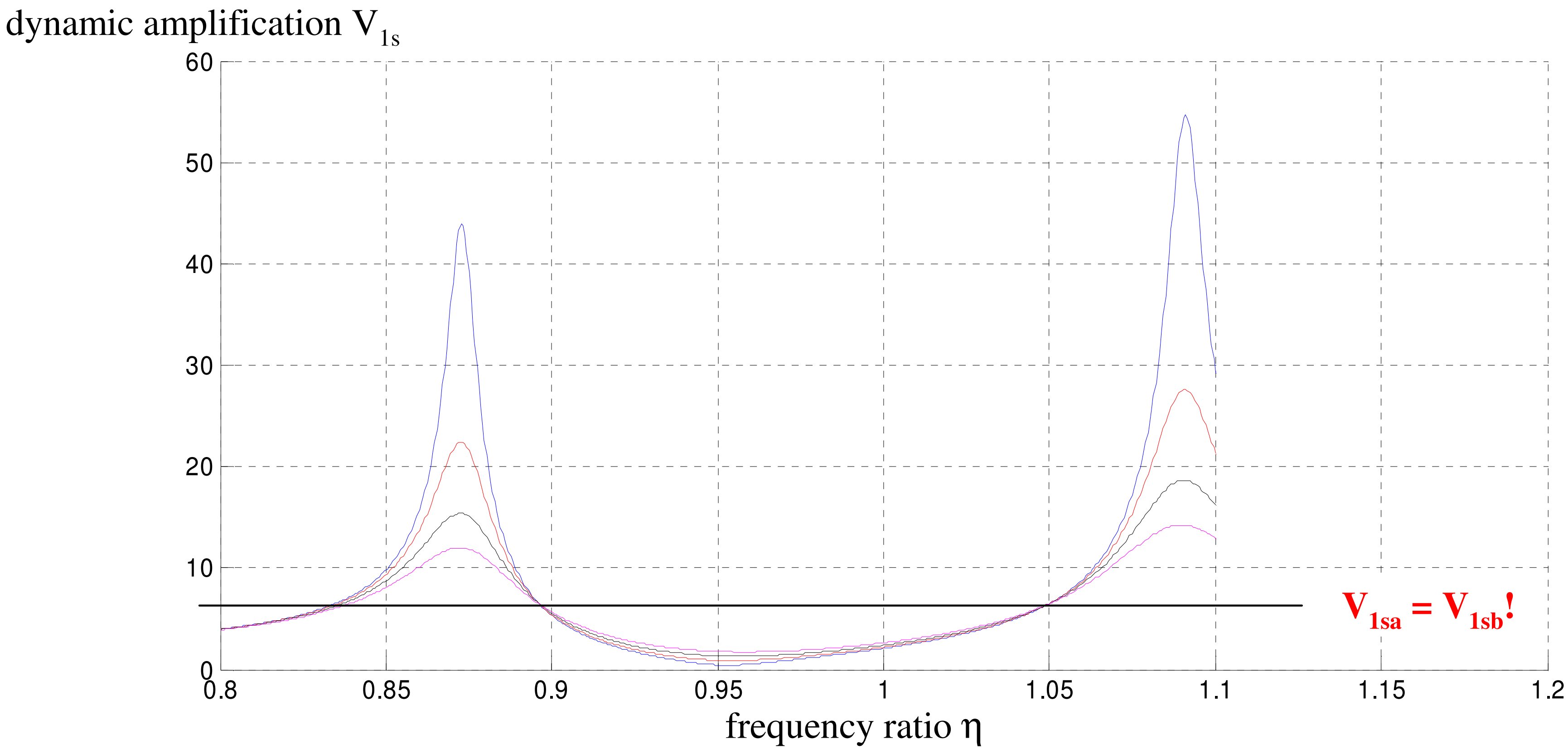
The robustness would increase if the sensitivity were not skewed. This can be achieved by requirering $V_{1sa} = V_{1sb}$. The resulting equation yields a condition for the tuning ratio κ:

$$\kappa_{opt} = \frac{\omega_d}{\omega_s} = \frac{1}{1+\mu}$$

The mass ratio m controls the split of the eigenfrequencies and therefore the width between the two peaks.

Resonance Spectrum for κ_{opt}

fixed parameters: μ = 0.05, ξ_s = 0.05, K_{opt} = 0.9524

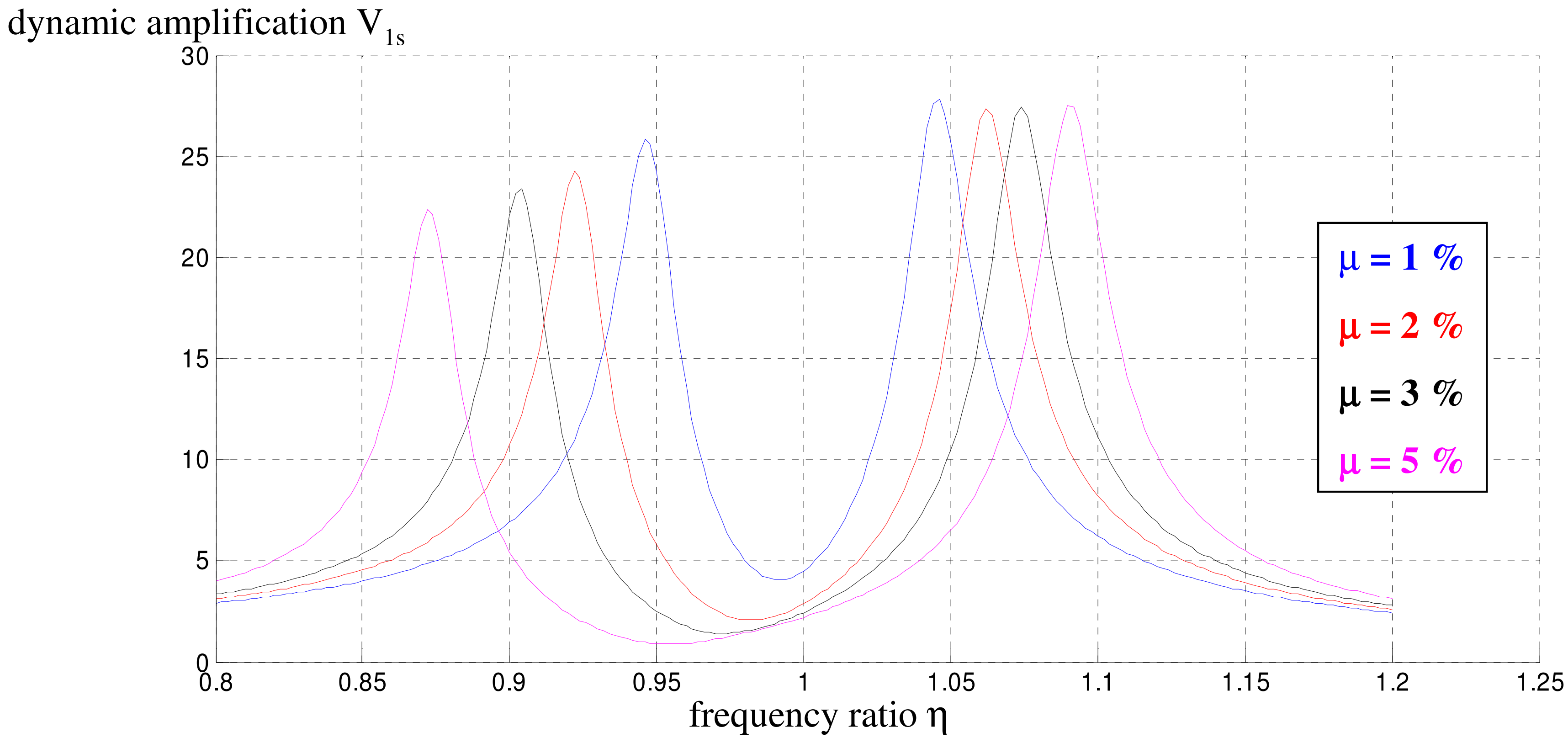


$$\eta^2_{a,b} = \frac{1}{1+\mu}\left\{1 \mp \sqrt{\frac{\mu}{2+\mu}}\right\}$$

$$V_{1sa} = V_{1sb} = \sqrt{\frac{2+\mu}{\mu}}$$

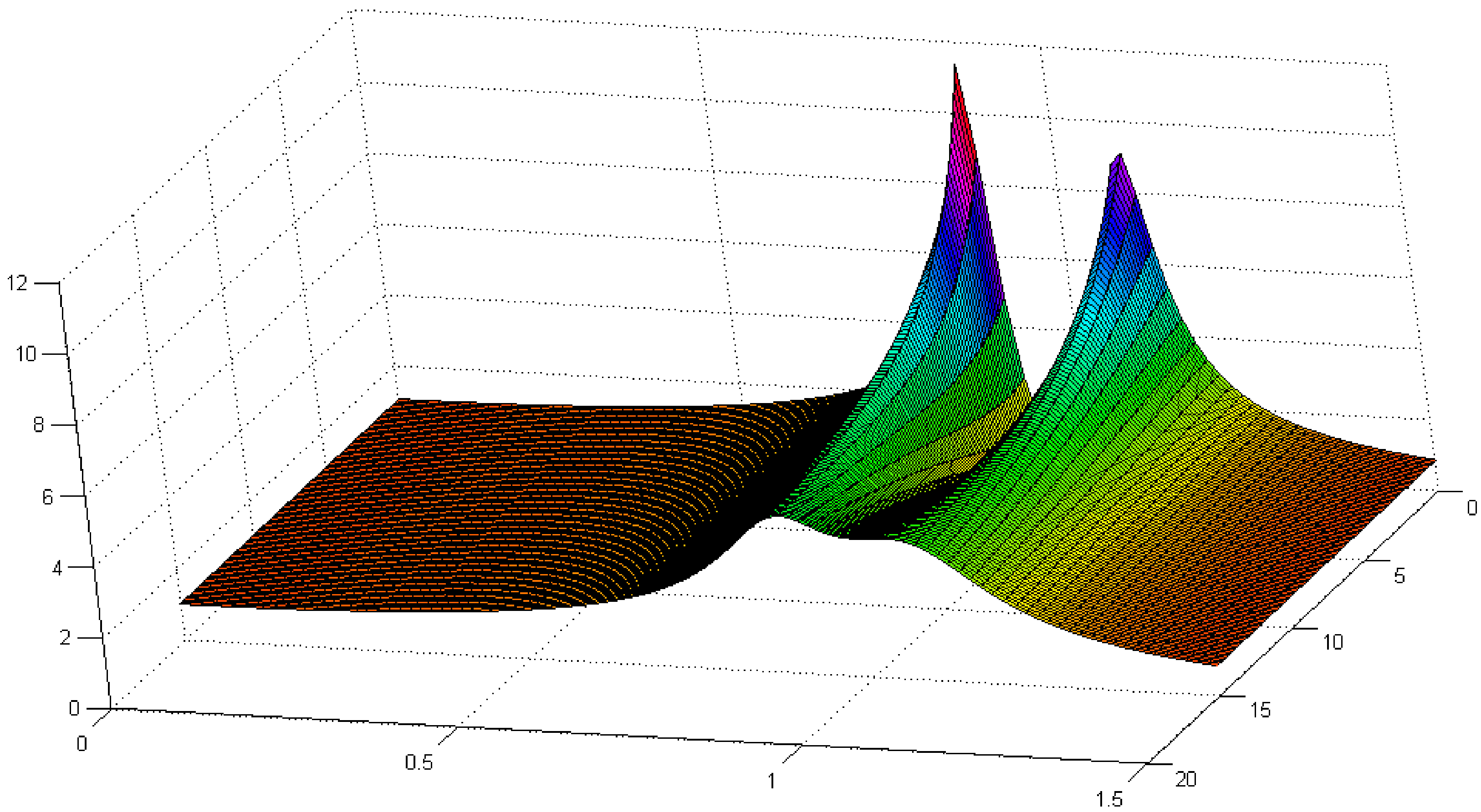
How to Find the Mass Ratio μ

fixed parameters: $\xi_s = 0.02$, $\kappa = \kappa_{opt}$



The plateau between the response peaks is widened by a larger μ which increases the bandwidth of effectiveness. Common are values of 1 % to 5 % of the *modal* mass for μ.

Observation 2: Effect of Damping ξ_d



The damping ξ_d of the TMD lowers the effectiveness, yet also decreases the sensitivity.

Requirement 2: Plateaulike Response

To achieve an almost constant resonance curve between the points A and B, we require that at an intermediate point C we have the same amplification as in A and B!

$$\eta_c^2 = \frac{1}{2}\left\{\eta_a^2 + \eta_b^2\right\} = \frac{1}{1+\mu}$$

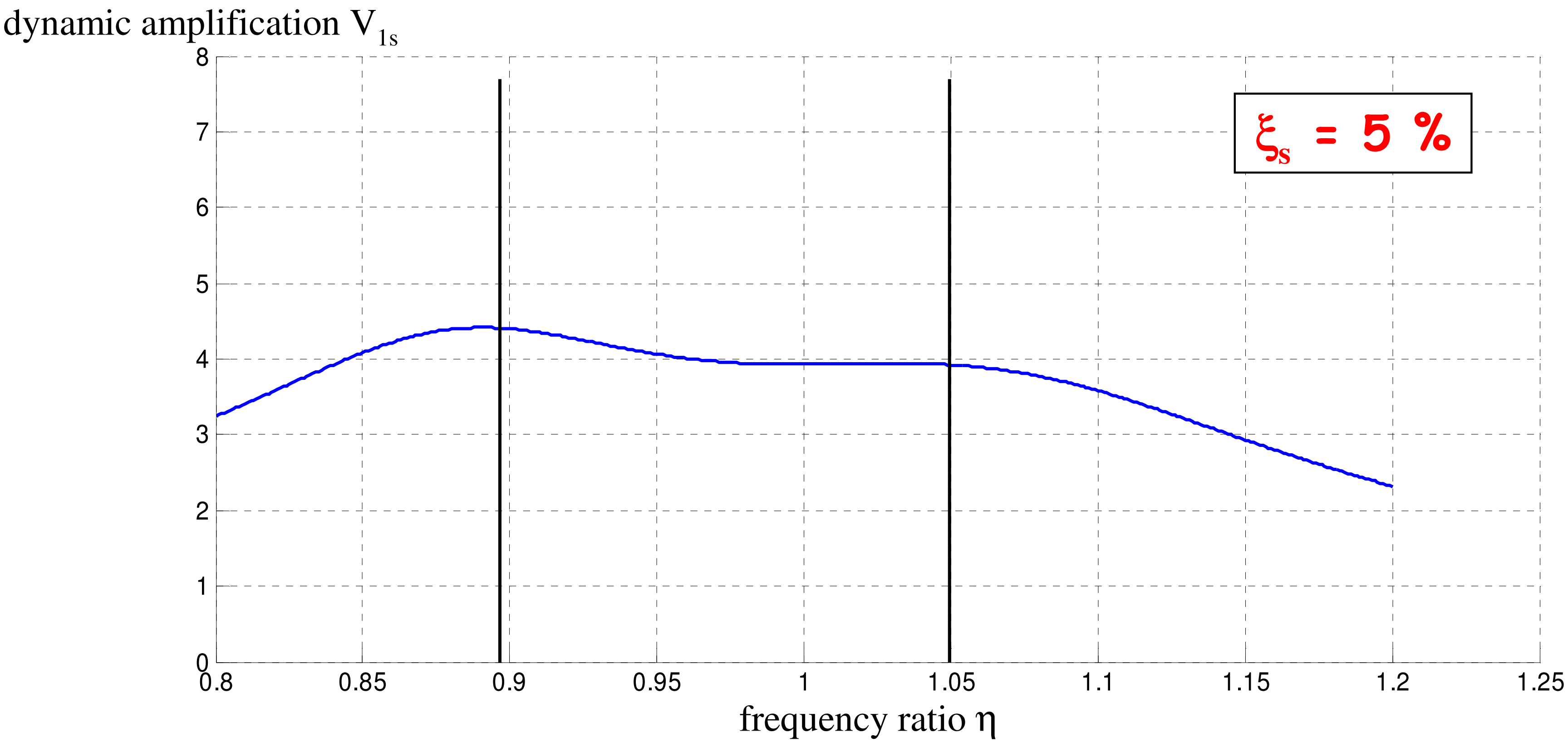
$$\xi_{opt} = \sqrt{\frac{\mu}{2(1+\mu)}}$$

The damping ξ_d of the TMD lowers the effectiveness, yet also decreases the sensitivity.

Optimum Damping ξ_{opt}

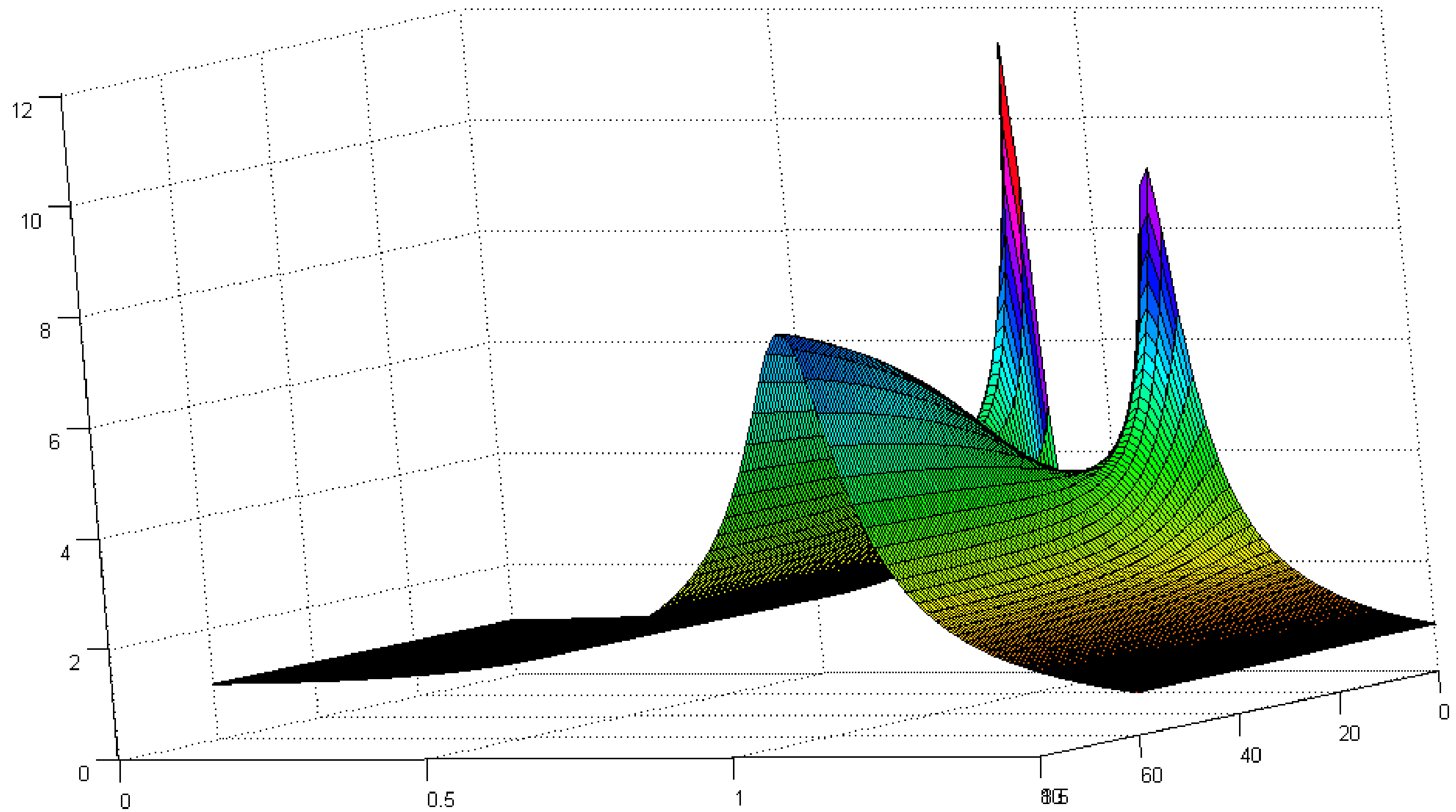
chosen **computed**

$$\mu = 0.05 \rightarrow \kappa_{opt} = 0.9524 \rightarrow \xi_{opt} = 0.1543$$



Observation 3: Overlarge Damping

An overlarge damping leads to a rigid coupling of the TMD to the structure and the response converges to that of an undamped structure. Overlarge damping must be avoided!



Alternative Optimization

Criteria by Den Hartog:

- impose a horizontal slope in point A
  - $\Rightarrow$ damping ratio ξ_a
- impose a horizontal slope in point B
  - $\Rightarrow$ damping ratio ξ_b
- mean value of ξ_a and ξ_b is ξ_{opt}

$$\xi_{opt} = \sqrt{\frac{3\mu}{8(1+\mu)}}$$

Optimization of Other Parameters

excitation	target	κ_{opt}	ξ_{opt}
harmonic force	s	$\frac{1}{1+\mu}$	$\sqrt{\frac{3\mu}{8(1+\mu)}}$
	$\ddot{s}$	$\sqrt{\frac{1}{1+\mu}}$	$\sqrt{\frac{3\mu}{4(2+\mu)}}$
harmonic base acceleration	s	$\sqrt{\frac{2-\mu}{2(1+\mu)^2}}$	$\sqrt{\frac{3\mu}{4(1+\mu)(2-\mu)}}$
	$\ddot{s}$	$\frac{1}{1+\mu}$	$\sqrt{\frac{3\mu}{8(1+\mu)}}$
stochastic load as white noise	σ_s^2	$\sqrt{\frac{2+\mu}{2(1+\mu)^2}}$	$\sqrt{\frac{\mu(4+3\mu)}{8(1+\mu)(2+\mu)}}$

from: Christian Petersen „Schwingungsdämpfer im Ingenieurbau", Maurer Söhne, 2001

Optimization for Complex Structures

A complex structure has several modes which can be excited at the same time. The load is not identical to the ones where analytical optimization criteria exist.

Apart from the amplitudes of the structure we must also know the the amplitudes of the TMD.

The optimization criteria allow a pre-design of the TMDs. Afterwards a more advanced time-domain analysis should be performed to compute the response (displacements, velocities, accelerations) of the coupled multi-mode TMD-structure system.

The time-domain algorithms should be able to model discrete damping elements.