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Lecture Series:

Structural Dynamics

Lecture 12:

Stochastic Excitation

Part C: Spectral Response of Complex Structures

Overview

- **Mathematical background**
- **SDOF-System under multiple loading**
  - **analytical solution for 2 correlated load processes**
  - **numerical example**
  - **generalization to an arbitrary number of loads**
- **MDOF-System: direct solution**
  - **theory**
  - **application**
- **MDOF-System: modal superposition**

Mathematical Background

Mathematical Review 1: Complex Numbers

Given: 2 complex numbers $\underline{z}_1$ and $\underline{z}_2$:

$$\underline{z}_1 = a + bi \quad \Rightarrow \quad \underline{\tilde{z}}_1 = a - bi$$

$$\underline{z}_2 = c + di \quad \Rightarrow \quad \underline{\tilde{z}}_2 = c - di$$

Define another complex number $\underline{z}$ as the product of $\underline{z}_1$ and $\underline{z}_2$:

$$\underline{z} = \underline{z}_1 \cdot \underline{z}_2 \quad \Rightarrow \quad \underline{z} = (ac - bd) + i(ad + bc)$$

Question: what is the conjugate of $\underline{z}$?

$$\underline{\tilde{z}} = (ac - bd) - i(ad + bc)$$

$$\underline{\tilde{z}}_1 \cdot \underline{\tilde{z}}_2 = (a - bi) \cdot (c - di) = (ac - bd) + i(-ad - bc) = \underline{\tilde{z}}$$

NB: the conjugate complex of a product is equal to the product of the individual conjugates.

Mathematical Review 2: FOURIER Transform

Given: 2 stochastic processes $f_1(t)$ and $f_2(t)$:

$$f_1(t) \rightarrow \underline{F}_1(i\Omega) = \int_{-\infty}^{+\infty} f_1(t) e^{-i\Omega t} dt$$

$$f_2(t) \rightarrow \underline{F}_2(i\Omega) = \int_{-\infty}^{+\infty} f_2(t) e^{-i\Omega t} dt$$

Question: FOURIER transform of a linear combination of f_1 and f_2?

$$g(t) = a_1 \cdot f_1(t) + a_2 \cdot f_2(t) \rightarrow \underline{G}(i\Omega) = \int_{-\infty}^{+\infty} [a_1 \cdot f_1(t) + a_2 \cdot f_2(t)] e^{-i\Omega t} dt$$

$$\Downarrow$$

$$\underline{G}(i\Omega) = a_1 \cdot \underline{F}_1(i\Omega) + a_2 \cdot \underline{F}_2(i\Omega)$$

NB: the FOURIER transform of a linear combination of stochastic processes is equal to the linear combination of the individual FOURIER transforms.

Mathematical Review 3: Cross-Spectra

Given: 2 stochastic processes $f_1(t)$ and $f_2(t)$ with $F_1(i\Omega)$ and $F_2(i\Omega)$:

$$\underline{F}_1(i\Omega) = a(\Omega) + b(\Omega)i \quad \Rightarrow \quad \tilde{\underline{F}}_1(i\Omega) = a(\Omega) - b(\Omega)i$$

$$\underline{F}_2(i\Omega) = c(\Omega) + d(\Omega)i \quad \Rightarrow \quad \tilde{\underline{F}}_2(i\Omega) = c(\Omega) - d(\Omega)i$$

The cross spectrum between processes $f_1(t)$ and $f_2(t)$ is given by:

$$S_{12}(i\Omega) = \lim_{T/2\to\infty} \frac{1}{T} \tilde{\underline{F}}_1(i\Omega) \cdot \underline{F}_2(i\Omega) \quad \Rightarrow \quad \underline{S}_{12}(i\Omega) = \lim_{T/2\to\infty} \frac{1}{T} \{(ac+bd) + i(ad-bc)\}$$

The cross spectrum between processes $f_2(t)$ and $f_1(t)$ is given by:

$$S_{21}(i\Omega) = \lim_{T/2\to\infty} \frac{1}{T} \tilde{\underline{F}}_2(i\Omega) \cdot \underline{F}_1(i\Omega) \quad \Rightarrow \quad \underline{S}_{21}(i\Omega) = \lim_{T/2\to\infty} \frac{1}{T} \{(ac+bd) - i(ad-bc)\}$$

$$S_{21}(i\Omega) = \tilde{S}_{12}(i\Omega)$$

The matrix of cross spectra is conjugate complex.

$$\mathbf{S}^T(i\Omega) = \tilde{\mathbf{S}}(i\Omega)$$

Mathematical Review 4: Autospectrum I

Given: 2 stochastic processes $f_1(t)$ and $f_2(t)$:

$f_1(t) \rightarrow \underline{F}_1(i\Omega)$

$f_2(t) \rightarrow \underline{F}_2(i\Omega)$

$$\rightarrow \underline{\mathbf{S}}_f(i\Omega) = \begin{bmatrix} S_{11}(\Omega) & \underline{S}_{12}(i\Omega) \\ \underline{S}_{21}(i\Omega) & S_{22}(\Omega) \end{bmatrix}$$

Question: autospectrum of a linear combination of f_1 and f_2?

$$g(t) = a_1 \cdot f_1(t) + a_2 \cdot f_2(t) \rightarrow S_g(\Omega) = \lim_{T/2\to\infty} \frac{1}{T} \underline{\tilde{G}}(i\Omega) \cdot \underline{G}(i\Omega)$$

$$\rightarrow S_g(\Omega) = \lim_{T/2\to\infty} \frac{1}{T} [(a_1 \cdot \tilde{F}_1 + a_2 \cdot \tilde{F}_2)(a_1 \cdot F_1 + a_2 \cdot F_2)]$$

$$\rightarrow S_g(\Omega) = \lim_{T/2\to\infty} \frac{1}{T}(a_1 a_1 \tilde{F}_1 F_1 + \lim_{T/2\to\infty} \frac{1}{T} a_1 a_2 \tilde{F}_1 F_2 + \lim_{T/2\to\infty} \frac{1}{T} a_2 a_1 F_1 \tilde{F}_2 + \lim_{T/2\to\infty} \frac{1}{T} a_2 a_2 \tilde{F}_2 F_2$$

Mathematical Review 4: Auto-Spectrum II

The 4 limits yield the 4 elements of the matrix of cross spectra:

$$S_g(\Omega) = a_1 a_1 S_{11}(\Omega) + a_1 a_2 \underline{S}_{12}(i\Omega) + a_2 a_1 \underline{S}_{21}(i\Omega) + a_2 a_2 S_{22}(\Omega)$$

In matrix form:

$$S_g = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} S_{11} & \underline{S}_{12} \\ \underline{S}_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

The cross spectra S_{12} and S_{21} are conjugate complex so that their imaginary parts cancel out and the autospectrum becomes real. We have a full coupling of all auto- and cross-spectra. The variance of the total response is the sum of the variances of the individual processes plus a coupling "cross variance" which stems from the integral of the cross-spectrum.

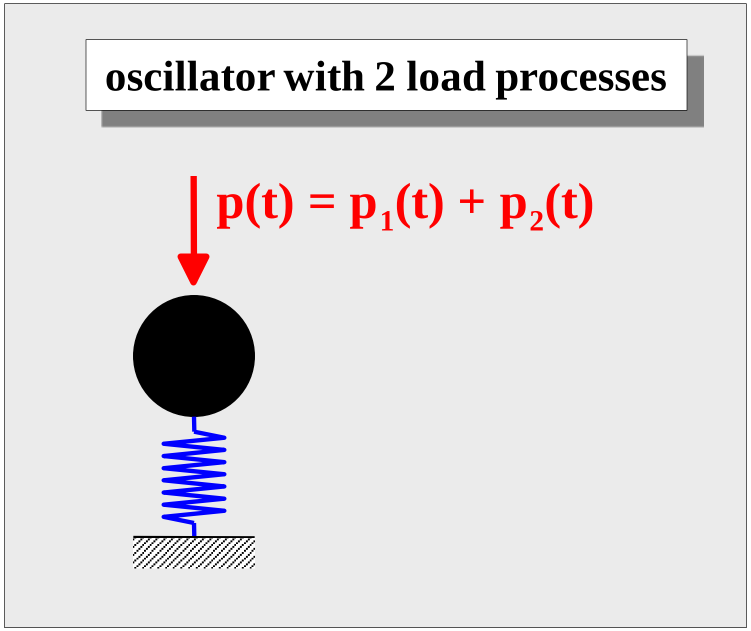
$$\sigma_g^2 = \int_\Omega S_g(\Omega)d\Omega = a_1 a_1 \sigma_{11}^2 + 2a_1 a_2 \sigma_{12}^2 + a_2 a_2 \sigma_{22}^2$$

$$\sigma_{12}^2 = \int_\Omega \mathrm{Re}(S_{12}(\Omega))d\Omega$$

Application I:
SDOF Oscillator with 2 Load Histories

analytical solution

Application I:
SDOF-System With 2 Load Processes



Problem:

What is σ^2 of the total response, caused by the total load $p_g = p_1 + p_2$?

Given:

Two independent load processes with zero mean:

- auto-spectra S_{p11} and S_{p22}
- cross-spectra S_{p12} and S_{p21}

The processes are independent but correlated!

System properties:

Time domain:

- k, m, c.

Frequency domain:

- complex transfer function $H(i\Omega)$.

Time Domain Solution

Solutions for the loads $p_1(t)$ and $p_2(t)$:

$p_1(t)$ → **solution of the equation of motion** → $v_1(t)$ → σ^2_{v1t} v_{1max}

$p_2(t)$ → **solution of the equation of motion** → $v_2(t)$ → σ^2_{v2t} v_{2max}

The SDOF-oscillator is linear: the superposition principle holds!

$p_g(t) = p_1(t) + p_2(t)$ → **superposition** → $v_g(t) = v_1(t) + v_2(t)$ → σ_{vgt}

Neither the maximum value nor the variance of the total response is equal to the sum of the individual variances!

$$\left|v_{g\,max}\right| \neq \left|v_{1max}\right| + \left|v_{2\,max}\right|$$

$$\sigma^2_{vgt} \neq \sigma^2_{v1t} + \sigma^2_{v2t}$$

$$\sigma_{vgt} \neq \sigma_{v1t} + \sigma_{v1t}$$

Spectral Matrix of the Individual Solutions

The FOURIER transforms of the solutions v_1 and v_2 can be calculated in the frequency domain from the FOURIER transforms of the 2 load processes p_1 and p_2 with the mechanical transfer function $\underline{H}$:

$v_1(t)$ $\underline{V}_1(i\Omega) = \underline{H}(i\Omega) \cdot \underline{P}_1(i\Omega)$

$v_2(t)$ $\underline{V}_2(i\Omega) = \underline{H}(i\Omega) \cdot \underline{P}_2(i\Omega)$

The spectral matrix of the responses v_1 and v_2 are then given by:

$$S_{v11} = \lim_{T/2\to\infty} \frac{1}{T} \underline{\tilde{V}}_1 \cdot \underline{V}_1 = \lim_{T/2\to\infty} \frac{1}{T} \underline{\tilde{H}}\,\underline{\tilde{P}}_1 \cdot \underline{H}\,\underline{P}_1 = \underline{\tilde{H}}\,\underline{H} S_{p11} = \chi_{vp} S_{p11}$$

$$S_{v22} = \underline{\tilde{H}}\,\underline{H} S_{p22} = \chi_{vp} S_{p22}$$

$$\underline{S}_{v12} = \underline{\tilde{H}}\,\underline{H}\,\underline{S}_{p12} = \chi_{vp} \underline{S}_{p12}$$

$$\underline{S}_{v21} = \underline{\tilde{H}}\,\underline{H}\,\underline{S}_{p21} = \chi_{vp} \underline{S}_{p21}$$

In matrix form:

$$S_v = \begin{bmatrix} S_{v11} & \underline{S}_{v12} \\ \underline{S}_{v21} & S_{v22} \end{bmatrix} = \begin{bmatrix} \underline{\tilde{H}} & 0 \\ 0 & \underline{\tilde{H}} \end{bmatrix} \begin{bmatrix} S_{p11} & \underline{S}_{p12} \\ \underline{S}_{p21} & S_{p22} \end{bmatrix} \begin{bmatrix} \underline{H} & 0 \\ 0 & \underline{H} \end{bmatrix}$$

Variance of the Total Response

We integrate the spectral matrix of the displacements over the entire frequency domain to get the individual variances:

$$\underline{S}_v = \begin{bmatrix} S_{v11} & \underline{S}_{v12} \\ \underline{S}_{v21} & S_{v22} \end{bmatrix} \;\Rightarrow\; \sigma^2_{v11} = \int_\Omega S_{v11}(\Omega)\,d\Omega \qquad \sigma^2_{v22} = \int_\Omega S_{v22}(\Omega)\,d\Omega$$

$$\sigma^2_{v12} = \int_\Omega \mathrm{Re}(\underline{S}_{v12}(\Omega))\,d\Omega$$

The variance of the total displacement can be calculated with to:

$$v_g(t) = v_1(t) + v_2(t) \;\Rightarrow\; \sigma^2_{vg} = \sigma^2_{v11} + 2\sigma^2_{v12} + \sigma^2_{v22}$$

$$v_{g\,max} = g \cdot \sigma_{vg}$$

g: a suitably chosen peak factor.

The sum for the total variance represents a complete combination of all elements of the spectral matrix where the combination factors are set to unity.

Special Cases I

Case 1: perfect positive correlation

$$v_1(t) = v_2(t) \rightarrow v_g(t) = 2v_1(t) \rightarrow \sigma_{vgt}^2 = 4\sigma_{v1}^2$$

$$S_v = \begin{bmatrix} S_{v11} & S_{v11} \\ S_{v11} & S_{v11} \end{bmatrix} \rightarrow \sigma_{vgs}^2 = \sigma_{v11}^2 + 2\sigma_{v11}^2 + \sigma_{v11}^2 \rightarrow \sigma_{vgs}^2 = 4\sigma_{v1}^2$$

Case 2: perfect negative correlation

$$v_1(t) = -v_2(t) \rightarrow v_g(t) = 0 \rightarrow \sigma_{vgt}^2 = 0$$

$$S_v = \begin{bmatrix} S_{v11} & -S_{v11} \\ -S_{v11} & S_{v11} \end{bmatrix} \rightarrow \sigma_{vgs}^2 = \sigma_{v11}^2 - 2\sigma_{v11}^2 + \sigma_{v11}^2 \rightarrow \sigma_{vgs}^2 = 0$$

Special Cases II

Case 3: complete de-correlation

$$v_1(t) \perp v_2(t) \rightarrow v_g(t) = v_1(t) + v_2(t) \rightarrow \sigma^2_{vgt} = ?$$

$$S_v = \begin{bmatrix} S_{v11} & "0" \\ "0" & S_{v22} \end{bmatrix} \rightarrow \sigma^2_{vgs} = \sigma^2_{v11} + \sigma^2_{v22}$$

We have seen that the cross-spectrum is not zero in the case of uncorrelated processes. The real part, however, fluctuates randomly about zero so that its integral over the frequency range becomes zero.

In the case of complete de-correlation we have an SRSS superposition of the root mean square! This leads to the SRSS scheme for arbitrary variables if we assume a *uniform peak factor* for all contributing processes.

$$\sigma_A = \frac{A_{max}}{g_A} \qquad \sigma_B = \frac{B_{max}}{g_B}$$

$$C(t) = A(t) + B(t) \rightarrow \sigma^2_C = \sigma^2_A + \sigma^2_V \rightarrow \sigma^2_C = \frac{A^2_{max}}{g^2_A} + \frac{B^2_{max}}{g^2_B} = \frac{C^2_{max}}{g^2_C}$$

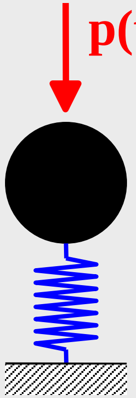
$$g_A = g_B = g_C \rightarrow A^2_{max} + B^2_{max} = C^2_{max}$$

Application I:
SDOF Oscillator with 2 Load Histories
numerical simulation

Numerical Example: SDOF-Oscillator

oscillator with 2 load processes

$p(t) = p_1(t) + p_2(t)$



Chosen:

$$k = 100\,\frac{kN}{m}$$

$$m = \mathbf{0.0396}\ \text{to}$$

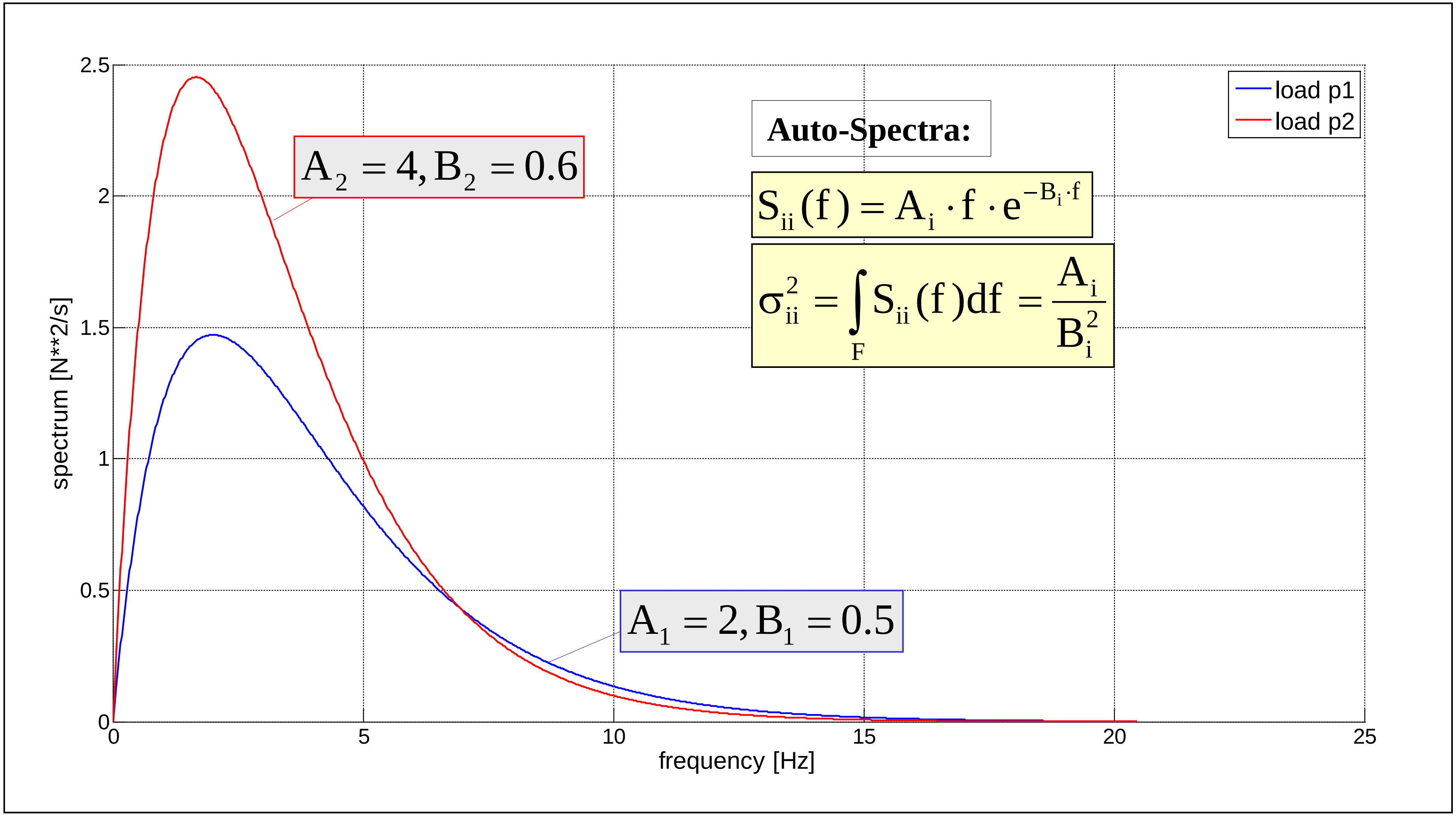
$$\xi = 0.05$$

$$\omega = \sqrt{\frac{k}{m}} = \mathbf{50.27}\,\frac{rad}{s}$$

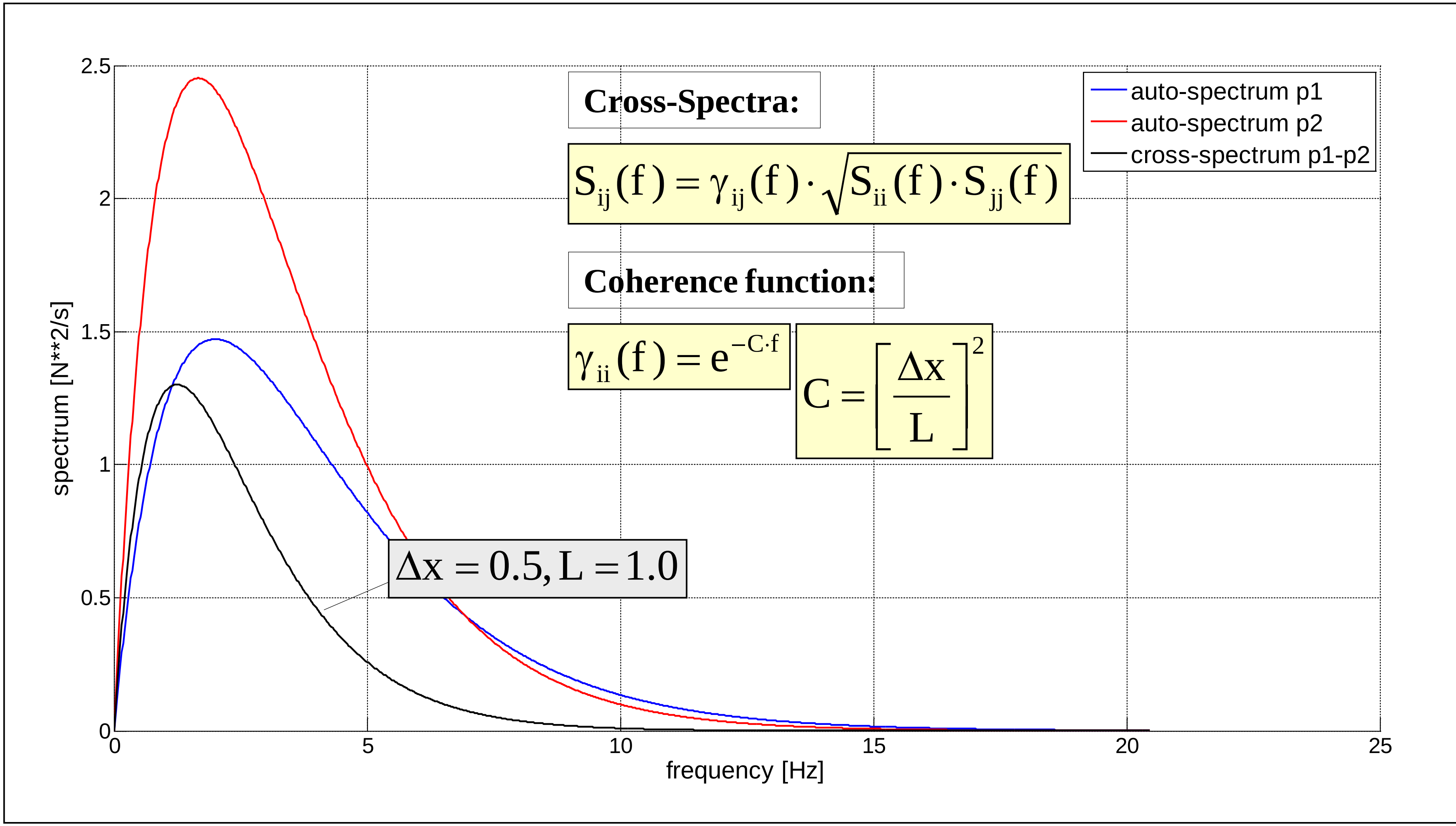
$$f = \frac{\omega}{2\pi} = 8.0\,Hz$$

$$T = \frac{1}{f} = 0.125\,s$$

Numerical Example: Auto-Spectra of the Loads

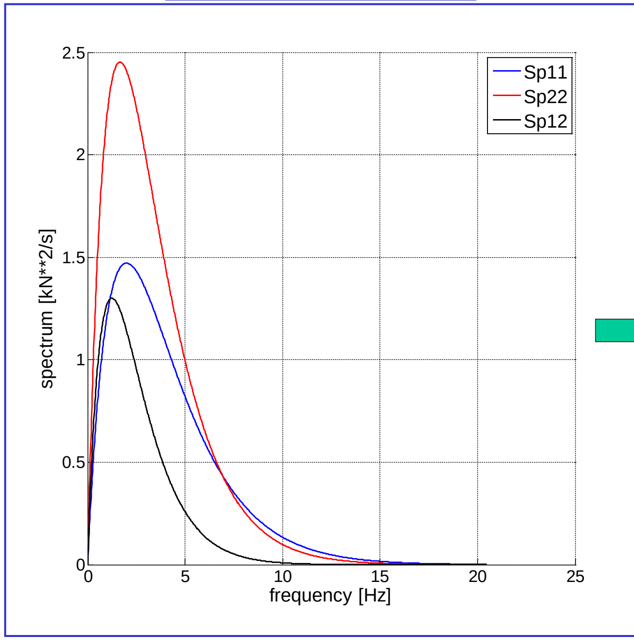


Numerical Example: Cross-Spectra of the Load

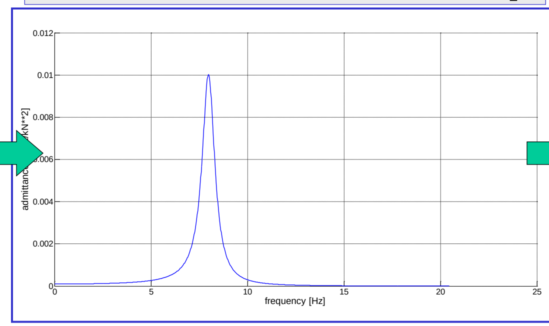


Spectral Solution for the Analytical Load Spectra

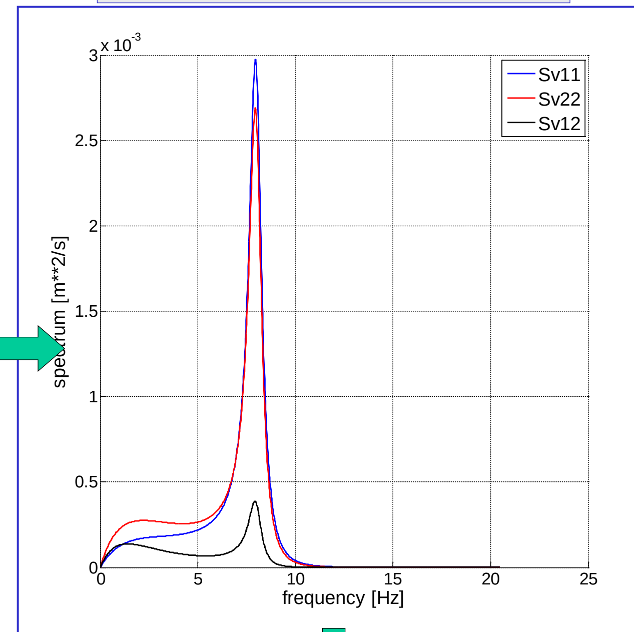
load spectra S_p



mechanical admittance χ_{vp}



displacement spectra S_p



$$\sigma_{vg}^2 = \int_\Omega S_{v11}(\Omega)d\Omega + \int_\Omega S_{v22}(\Omega)d\Omega + 2\int_\Omega \mathrm{Re}(S_{v12}(\Omega))d\Omega$$

Results for Different Levels of Correlation

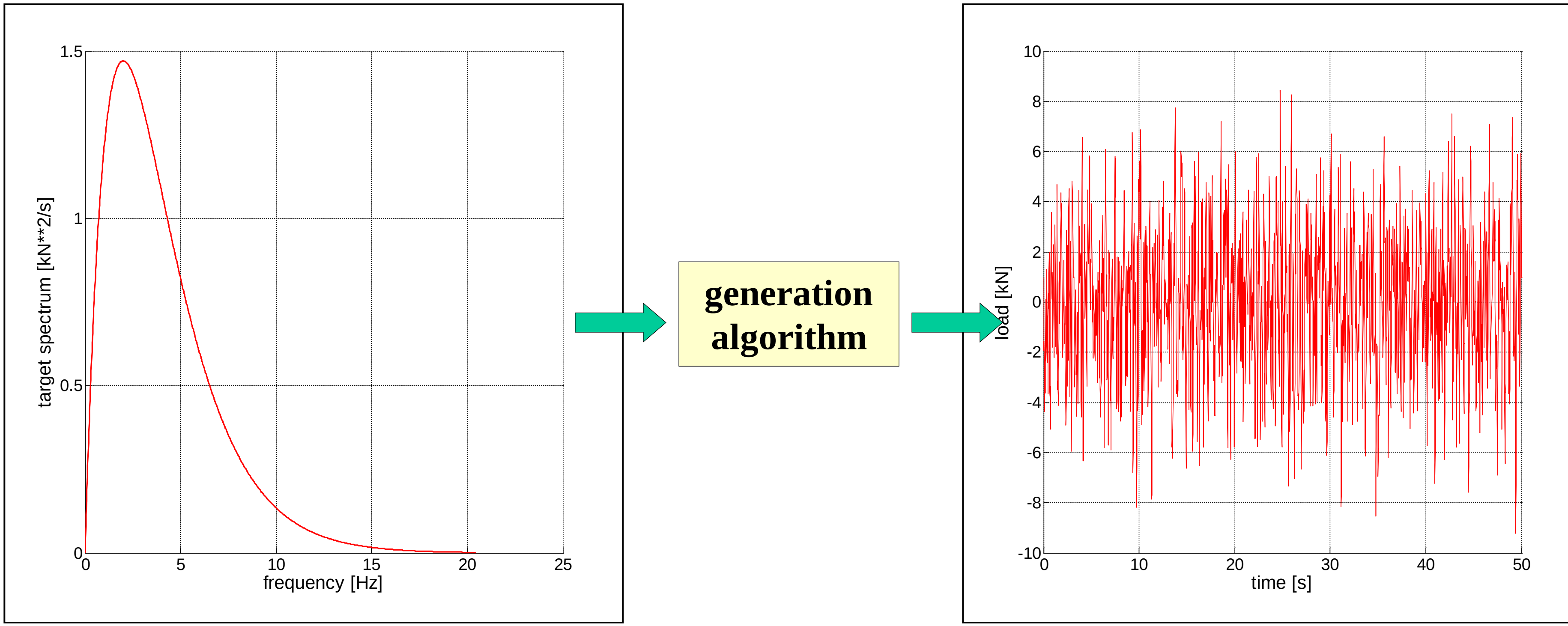
almost fully correlated (→ 0.05)

almost de-correlated (→ 2.0)

variance [mm²]	Δx/L				
	0.05	0.20	0.50	1.0	2.0
$Var(v_1)$	4.48	4.48	4.48	4.48	4.48
$Var(v_2)$	4.64	4.64	4.64	4.64	4.64
$Var(v_g)$	18.0	16.1	11.2	9.38	9.14
$Var(v_1) + Var(v_2)$	9.12	9.12	9.12	9.12	9.12

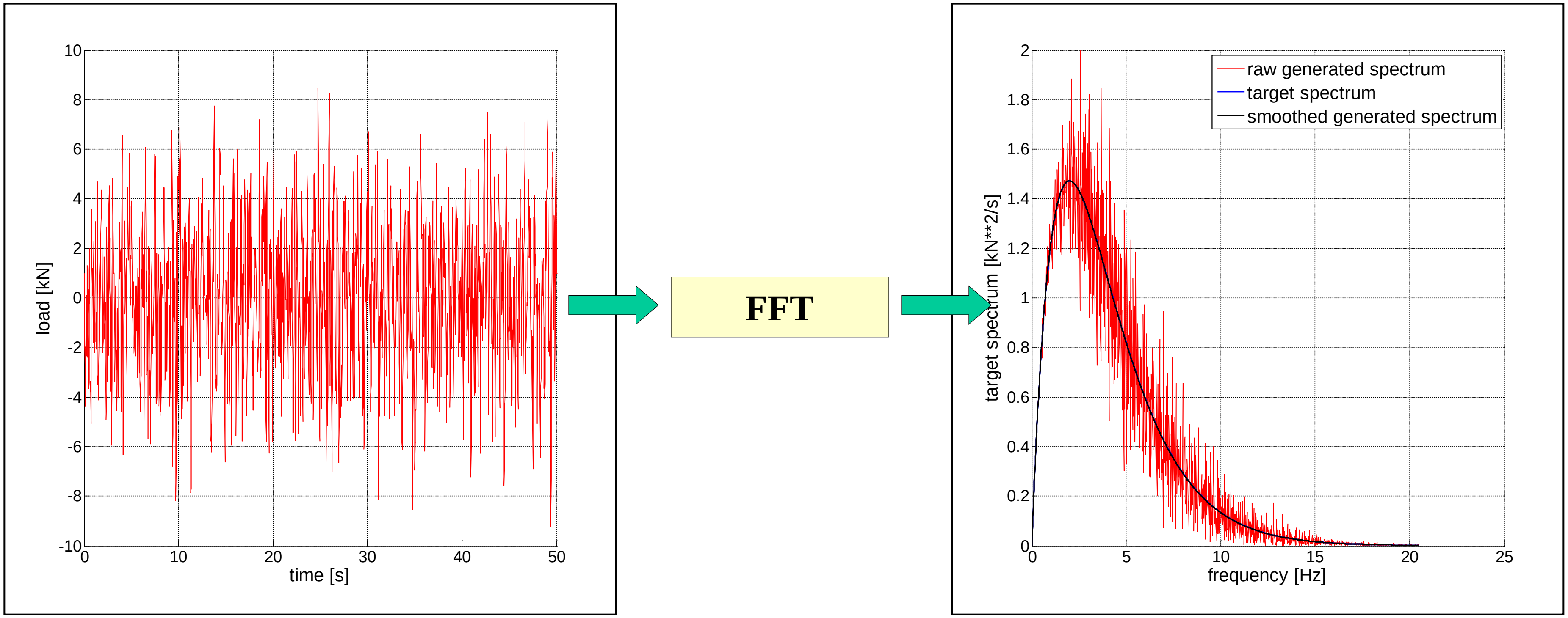
Time Domain Analysis: Generation of Realizations

Stochastic loads are defined in the spectral domain. For a time domain analysis we must generate numerical samples – realizations – of the stochastic process. Such samples do not capture the stochastic process in its entirety due to their limited length and due to numerical errors introduced in the generation process. There are different methods, here used is the WAWS (WEIGHTED AMPLITUDE WAWE SUPERPOSITION.



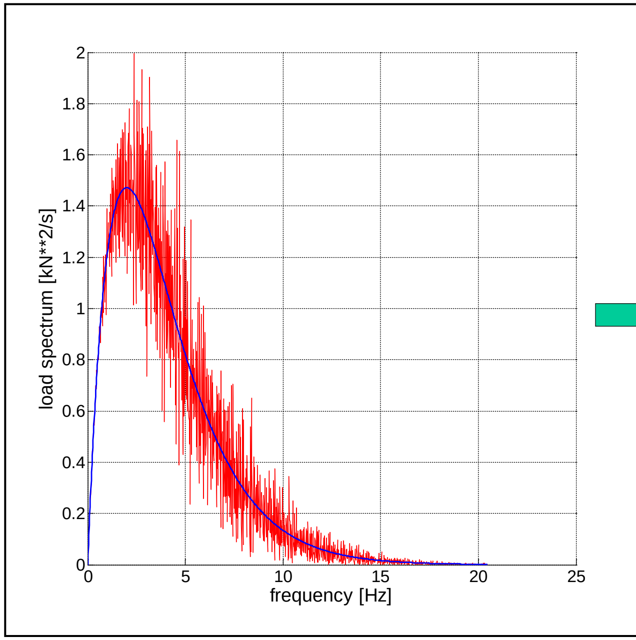
Check of the Realizations

From the generated sample we can compute its spectral density and compare it to the target spectrum. Typical for generated or measured time histories is a more or less pronounced fluctuation about the "true" spectrum. The oscillating spectrum can be smoothed by fitting a suitable curve through the data points. Since here we know the mathematical shape of the target spectrum, we can perform a nonlinear parameter fit for A and B and find that the smoothed spectrum coincides almost perfectly with the target spectrum.

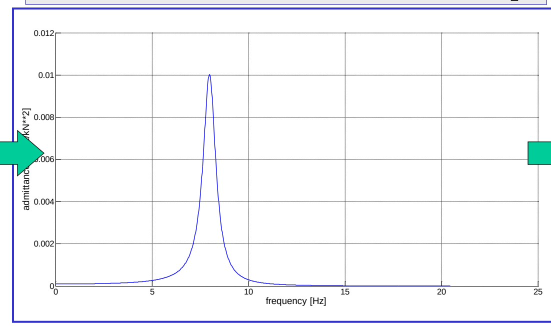


Spectral Solution for the Generated Load Spectra

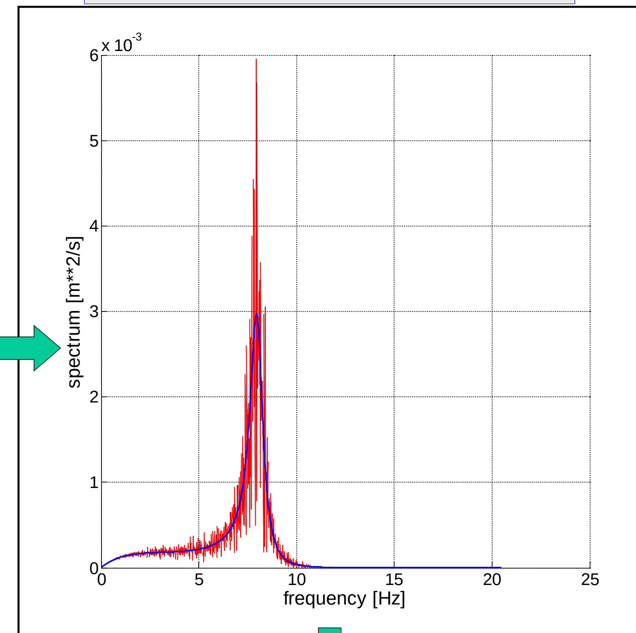
load spectra S_p



mechanical admittance χ_{vp}



displacement spectra S_v



$$\sigma_{vg}^2 = \int_\Omega S_{v11}(\Omega)d\Omega + \int_\Omega S_{v22}(\Omega)d\Omega + 2\int_\Omega \mathrm{Re}(S_{v12}(\Omega))d\Omega$$

Results for Different Levels of Correlation

almost fully correlated (→ 0.05)

almost de-correlated (→ 2.0)

The spectral approach is very stable!

blue: analytical spectra
black: generated spectra

variance [mm^2]	Δx/L				
	0.05	0.20	0.50	1.0	2.0
$Var(v_1)$	4.48 4.48	4.48 4.48	4.48 4.48	4.48 4.48	4.48 4.48
$Var(v_2)$	4.64 4.62	4.64 4.59	4.64 4.63	4.64 4.64	4.64 4.64
$Var(v_g)$	18.0 18.0	16.1 16.0	11.2 11.1	9.38 9.29	9.14 9.05
$Var(v_1) + Var(v_2)$	9.11 9.10	9.11 9.07	9.11 9.10	9.11 9.11	9.11 9.12

Generation of Time Histories of the Load I

WAWS algorithm: sum of harmonic components

$$p_i(t) = 2\sqrt{\Delta\omega}\sum_{j=1}^{i}\sum_{k=1}^{N}\left|L_{jk}(\omega_k)\right|\cdot\cos[(\omega_k t + \varphi_{jk})$$

$$\mathbf{L}(\omega)\cdot\mathbf{L}^T(\omega) = \mathbf{S}(\omega)$$

L: lower triangular matrix

Frequency domain

The maximum frequency (parameter N) and the frequency increment Δf of the generation have been chosen such that the area under the auto-spectrum is captured with almost perfect accuracy.

$$f_{max} = 1023\cdot\Delta f = 20.46\,\text{Hz}$$

$$\Delta f = 0.02\,\text{Hz}$$

$$\sigma_p^2 = \int_F S(f)df = \int_F A\cdot f\cdot e^{-Bf}df = \frac{A}{B^2}$$

	variance			
	A/B^2	**S_{target}**	**S_{gen}**	**p(t)**
p_1	**8.000**	**7.997**	**8.000**	**8.004**
p_2	**11.111**	**11.110**	**11.072**	**11.077**

The load histories capture the areas under the target spectra almost perfectly.

Generation of Time Histories of the Load II

Time domain

$$\Delta f = 0.02\,\text{Hz} \quad \Rightarrow \quad T_{max} = \frac{1}{\Delta f} = 50\,\text{s}$$

The time history would repeat after T_{max} = 50 s.

Time histories of the load have been generated with 2048 steps each, so that the Nyquist frequency of the time history becomes equal to the maximum frequency of the generation. We face two difficulties:

- **The time histories have finite length.**
- **The time step for the integration of the SDOF-oscillator must be sufficiently small.**

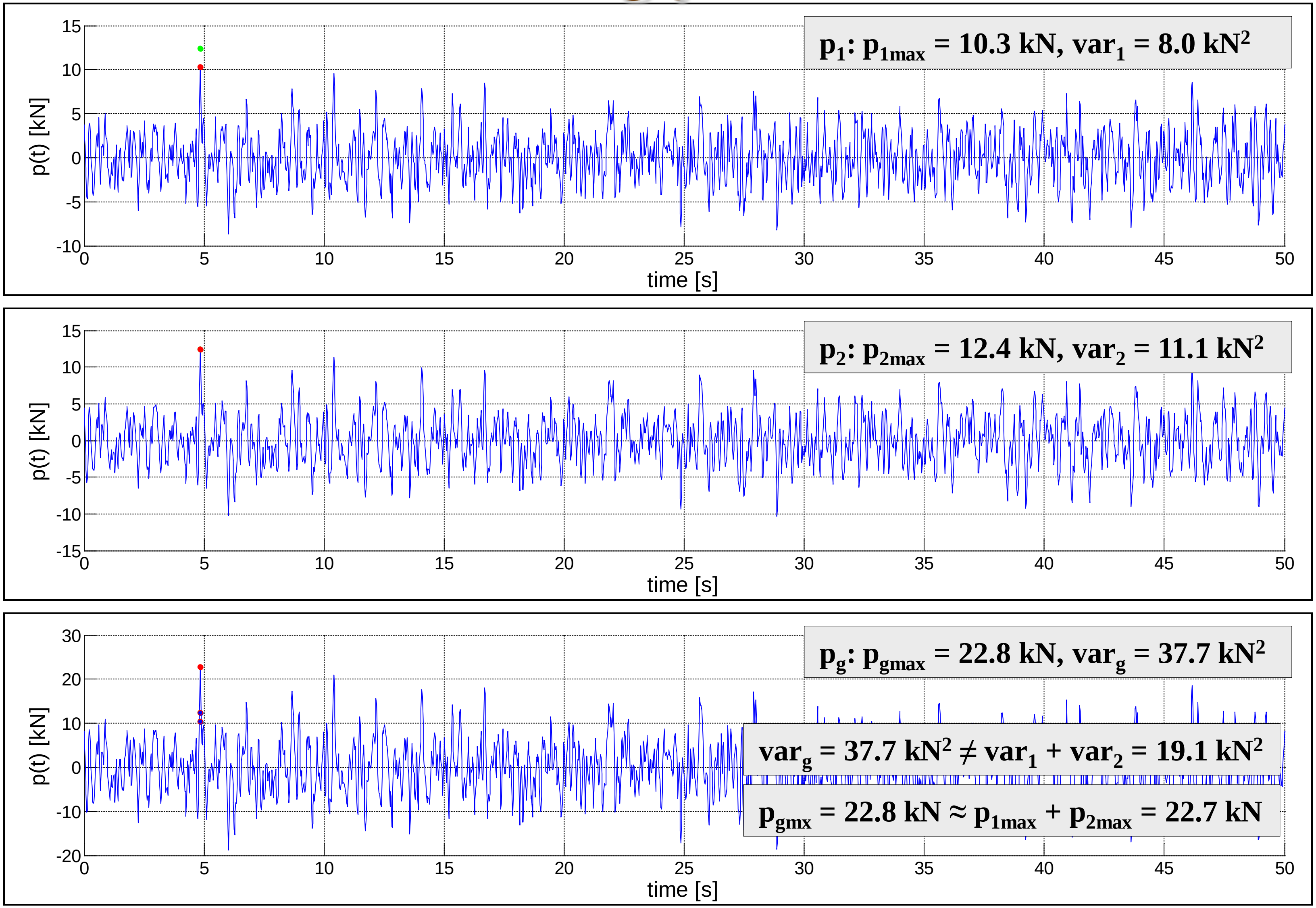
$$\Delta t_{gen} = \frac{T_{max}}{2047} = 0.0244\,\text{s} \approx \frac{T}{5}$$

The generation time step Δt_{gen} is too coarse!

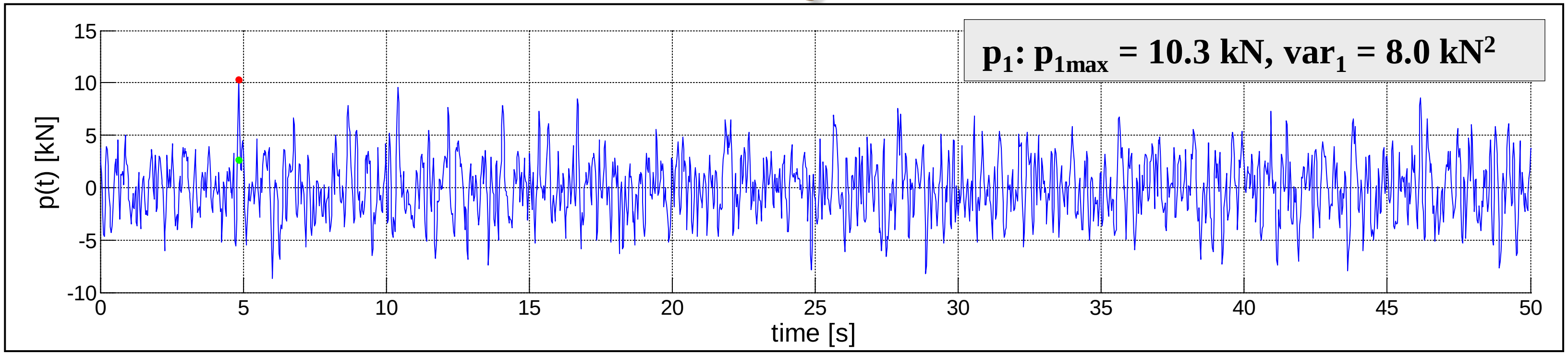
$$\Delta t_{ana} = \frac{\Delta t_{gen}}{16} \approx \frac{T}{80} \quad \Rightarrow \quad N_{step} = 16 \cdot N_{gen} = 32768$$

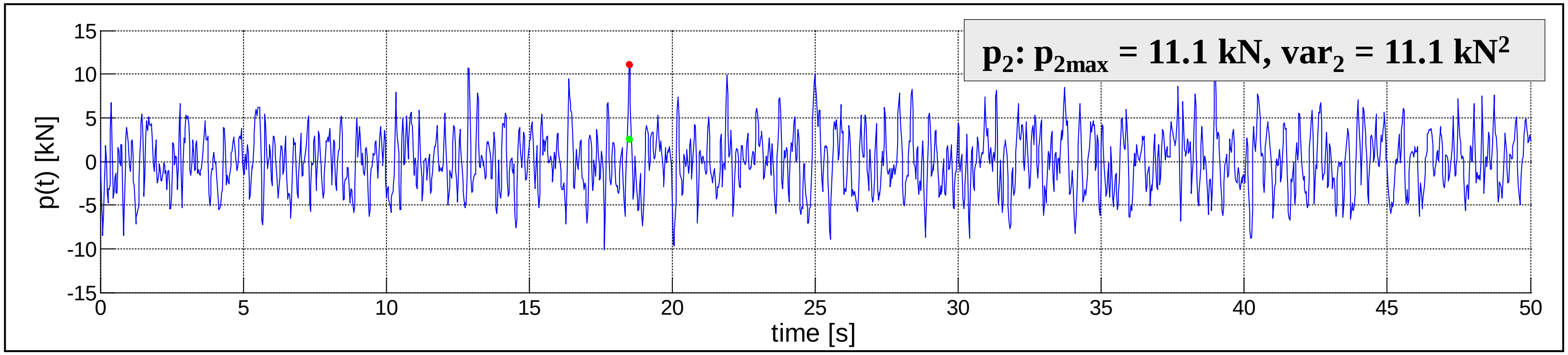
The finer time axis only helps in the time integration, it adds no additional information to the time history.

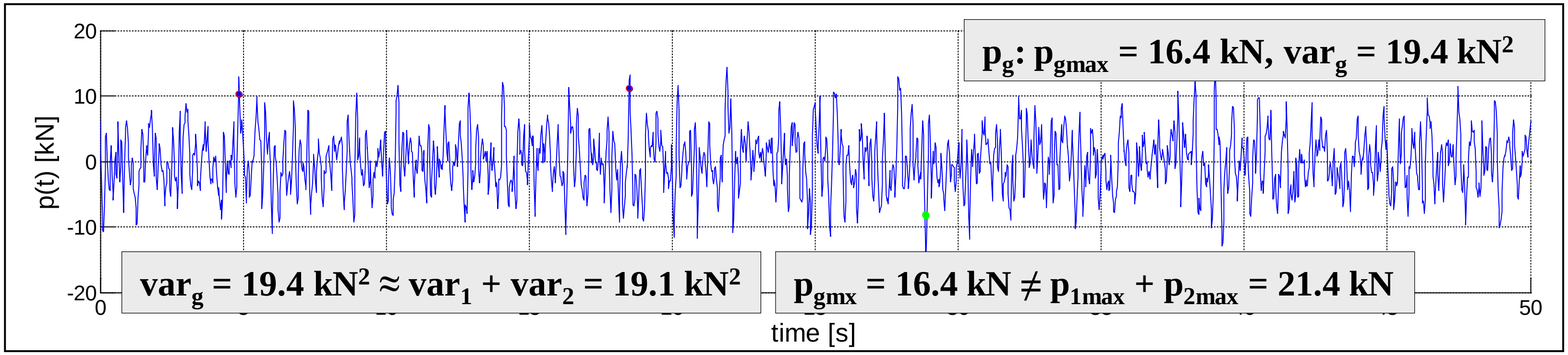
Load Histories I: Strongly Correlated – $\Delta x = 0.05$



Load Histories II: Weakly Correlated – Δx = 2.00

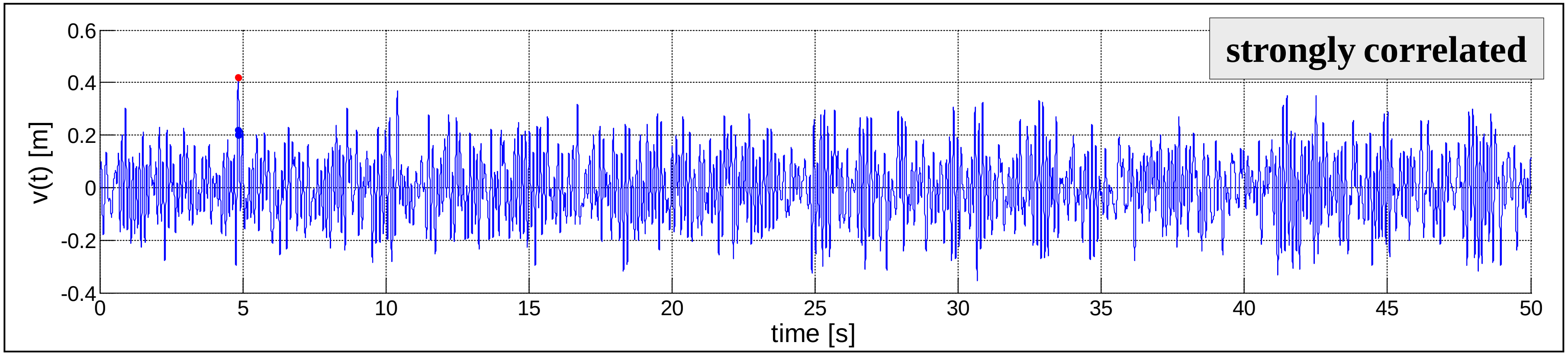


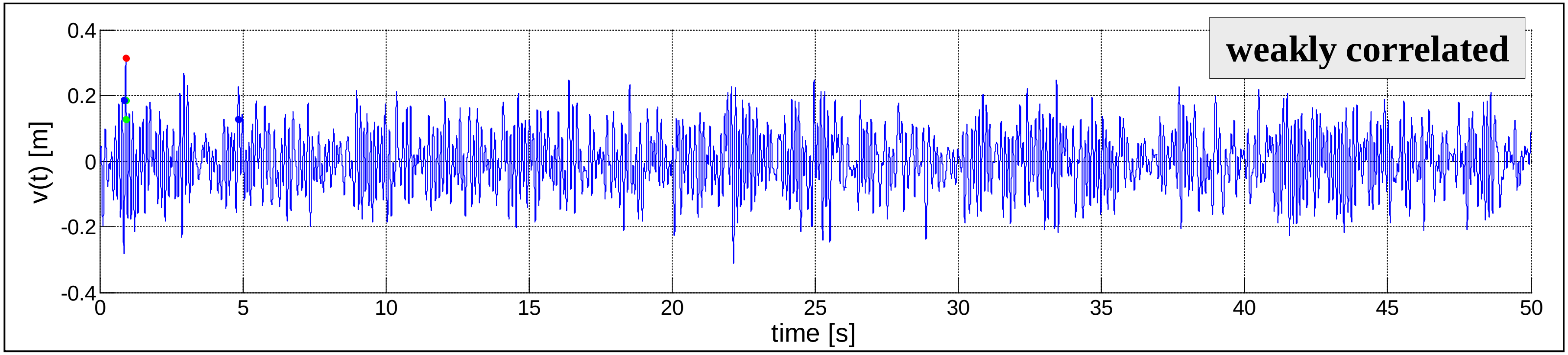




Displacement Histories

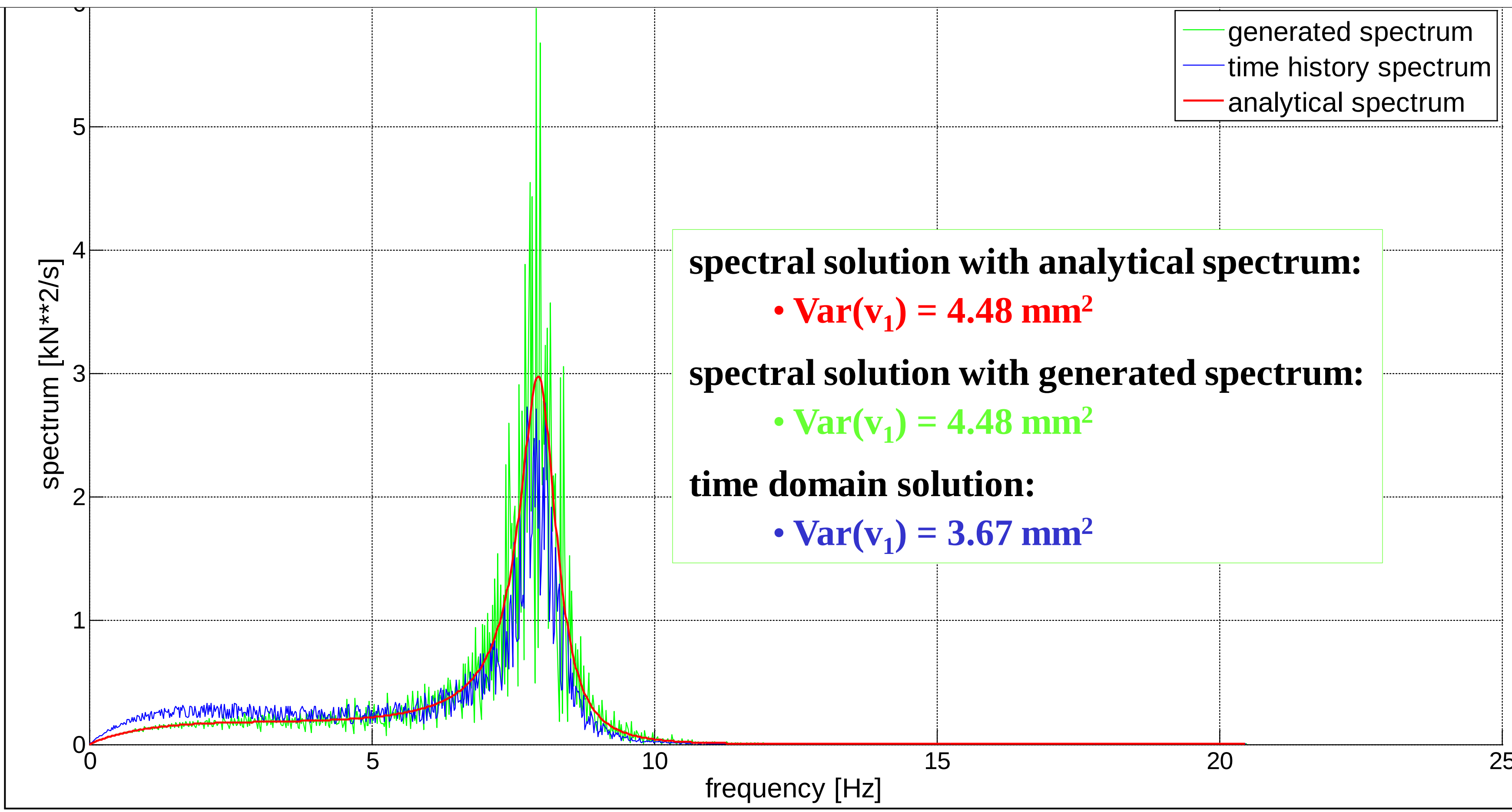
The displacement histories show the same behavior as the load histories. The maximum displacement of the strongly correlated response is equal to the sum of the individual maxima. Growing de-correlation reduces the response – for the weakly correlated response we have again the total variance as the sum of the individual variances.





Strongly Correlated: Plot of S_{v11}

The spectrum from the time history of v(t) is not identical to the spectrum computed in the spectral domain from load spectrum. When checking the resulting variances we find: The fluctuations of the load spectrum did not noticeably deteriorate the solution, but the time integration does so!



spectral solution with analytical spectrum:

- **$Var(v_1) = 4.48\ mm^2$**

spectral solution with generated spectrum:

- **$Var(v_1) = 4.48\ mm^2$**

time domain solution:

- **$Var(v_1) = 3.67\ mm^2$**

Time History Analysis vs. Spectral Analysis

A ***time history analysis*** based on given ***spectral loads*** requires the generation of ***numerical samples*** of the stochastic process. These samples carry only ***limited information*** regarding the true stochastic process. This reduced information content leads to a deterioration of the numerical results. The longer the time history is, the better can it capture the underlying stochastic process.

Different realizations give different results, the same as the rerun of an experiment would never exactly reproduce a previous experiment. The results of a time domain analysis based on a realization is itself a stochastic variable which would fluctuate about the true value. The entirety of all possible realizations would approach the true stochastic process. So the ***mean value*** of the variances from a large enough number of time domain analyses would converge to the true variance.

That means: time domain analyses are costly. We need a sufficient number of realizations to obtain trustworthy results. In wind engineering the typical number is 30.

A spectral analysis, on the other hand, is not subject to these fluctuations. One ***single analysis*** captures the true situation and is therefore sufficient. It does impose, however, large storage requirements, which necessitate special programming strategies when developing a finite element code in the spectral domain.

Application I:

SDOF Oscillator

Extension to Multiple Load Histories

An Arbitrary Number of Load Processes

We have an arbitrary number N of correlated load processes $p_k(t)$

$$p_g(t) = a_1 \cdot p_1(t) + a_2 \cdot p_2(t) + \cdots + a_N \cdot p_N(t) = \mathbf{A}^T \mathbf{p}(t)$$

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix} \qquad \mathbf{p}^T = \begin{bmatrix} p_1 & p_2 & \dots & p_N \end{bmatrix}$$

The load correlation is expressed in the spectral domain by the matrix of auto- and cross-spectra:

$$\underline{\mathbf{S}}_p = \begin{bmatrix} S_{p11} & \underline{S}_{p12} & \cdots & \underline{S}_{p1N} \\ \underline{S}_{p21} & S_{p22} & \cdots & \underline{S}_{p2N} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{S}_{pN1} & \underline{S}_{pN2} & \cdots & S_{pNN} \end{bmatrix}$$

Spectral Response of the Individual Responses

For each element of S_p we can compute the corresponding element of S_v:

$$\underline{\mathbf{S}}_v = \begin{bmatrix} S_{v11} & \underline{S}_{v12} & \cdots & \underline{S}_{v1N} \\ \underline{S}_{v21} & S_{v22} & \cdots & \underline{S}_{v2N} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{S}_{vN1} & \underline{S}_{vN2} & \cdots & S_{vNN} \end{bmatrix} = \chi_{vp} \begin{bmatrix} S_{p11} & \underline{S}_{p12} & \cdots & \underline{S}_{p1N} \\ \underline{S}_{p21} & S_{p22} & \cdots & \underline{S}_{p2N} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{S}_{pN1} & \underline{S}_{pN2} & \cdots & S_{pNN} \end{bmatrix}$$

An integration over the entire frequency range F yields the matrix of variances:

$$\boldsymbol{\sigma}_v^2 = \begin{bmatrix} \int_F S_{v11}\,df & \int_F \underline{S}_{v12}\,df & \cdots & \int_F \underline{S}_{v1N}\,df \\ \int_F \underline{S}_{v21}\,df & \int_F S_{v22}\,df & \cdots & \int_F \underline{S}_{v2N}\,df \\ \cdots & \cdots & \cdots & \cdots \\ \int_F \underline{S}_{vN1}\,df & \int_F \underline{S}_{vN2}\,df & \cdots & \int_F S_{vNN}\,df \end{bmatrix}$$

Total Response

The variance of the total response is computed as a complete combination of all elements of the matrix of variances, where the combination factors are equal to the combination factors of the load combination:

$$\sigma_v^2 = \begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix} \begin{bmatrix} \sigma_{v11}^2 & \underline{\sigma}_{v12}^2 & \cdots & \underline{\sigma}_{v1N}^2 \\ \underline{\sigma}_{v21}^2 & \underline{\sigma}_{v11}^2 & \cdots & \underline{\sigma}_{v2N}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \underline{\sigma}_{vN1}^2 & \underline{\sigma}_{vN2}^2 & \cdots & \sigma_{vNN}^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_N \end{bmatrix} = \mathbf{A}\boldsymbol{\sigma}_v^2\mathbf{A}^T$$

The complete combination can be expressed in compact form as a double sum. The total variation is real since the imaginary parts of the off-diagonal elements cancel out due to the fact that the off-diagonal elements are mutually conjugate complex.

$$\sigma_v^2 = \sum_{k=1}^{N}\sum_{j=1}^{N} a_k a_j \sigma_{vkj}^2$$

Direct Analysis of MDOF Systems in the Spectral Domain

General MDOF System

The equation of motion of a general system with N degrees of freedom is given by:

$$\mathbf{M} \cdot \ddot{\mathbf{v}} + \mathbf{C} \cdot \dot{\mathbf{v}} + \mathbf{K} \cdot \mathbf{v} = \mathbf{p}$$

The given loads and the unknown displacements, velocities and accelerations are all Nx1 matrices:

$$\mathbf{v} = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \cdots \\ v_N(t) \end{bmatrix} \quad \dot{\mathbf{v}} = \begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \cdots \\ \dot{v}_N(t) \end{bmatrix} \quad \ddot{\mathbf{v}} = \begin{bmatrix} \ddot{v}_1(t) \\ \ddot{v}_2(t) \\ \cdots \\ \ddot{v}_N(t) \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} p_1(t) \\ p_2(t) \\ \cdots \\ p_N(t) \end{bmatrix}$$

The stiffness, damping and mass matrices are all symmetric NxN matrices:

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & K_{22} & \cdots & K_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ K_{N1} & K_{N2} & \cdots & K_{NN} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & M_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} \end{bmatrix}$$

Response for Harmonic Loading

We assume a ***harmonic complex loading*** in the shape of:

$$\mathbf{p}(t) = \begin{bmatrix} \underline{\hat{p}}_1 \\ \underline{\hat{p}}_2 \\ \cdots \\ \underline{\hat{p}}_N \end{bmatrix} e^{i\Omega t} = \underline{\hat{\mathbf{p}}} \cdot e^{i\Omega t}$$

Each degree of freedom has a different ***complex load amplitude***, i.e. both ***load intensity*** and ***load phase*** can be different. The ***load frequency Ω***, however, is uniform for the entire load ensemble.

We know: the system will answer harmonically with the load frequency. Each degree of freedom has an unknown ***complex displacement amplitude***:

$$\mathbf{v}(t) = \begin{bmatrix} \underline{\hat{v}}_1 \\ \underline{\hat{v}}_2 \\ \cdots \\ \underline{\hat{v}}_N \end{bmatrix} e^{i\Omega t} = \underline{\hat{\mathbf{v}}} \cdot e^{i\Omega t}$$

$$\rightarrow \quad \dot{\mathbf{v}}(t) = \underline{\hat{\mathbf{v}}} \cdot i\Omega \cdot e^{i\Omega t} \quad \rightarrow \quad \ddot{\mathbf{v}}(t) = -\underline{\hat{\mathbf{v}}} \cdot \Omega^2 \cdot e^{i\Omega t}$$

Solution for Harmonic Loading

Equation of motion:

$$\ddot{\mathbf{v}}(t) = -\underline{\hat{\mathbf{v}}} \cdot \Omega^2 \cdot e^{i\Omega t} \qquad \dot{\mathbf{v}}(t) = \underline{\hat{\mathbf{v}}} \cdot i\Omega \cdot e^{i\Omega t} \qquad \mathbf{v}(t) = \underline{\hat{\mathbf{v}}} \cdot e^{i\Omega t} \qquad \mathbf{p}(t) = \underline{\hat{\mathbf{p}}} \cdot e^{i\Omega t}$$

$$\mathbf{M} \cdot \ddot{\mathbf{v}} + \mathbf{C} \cdot \dot{\mathbf{v}} + \mathbf{K} \cdot \mathbf{v} = \mathbf{p}$$

$$\left\{\mathbf{K} - \Omega^2 \mathbf{M} + i\Omega \mathbf{C}\right\} \underline{\hat{\mathbf{v}}} = \underline{\hat{\mathbf{p}}}$$

$$\underline{\mathbf{K}}(i\Omega) \cdot \underline{\hat{\mathbf{v}}}(i\Omega) = \underline{\hat{\mathbf{p}}}(i\Omega)$$

A ***complex, frequency-dependent stiffness matrix K*** connects the ***complex displacement amplitude*** with the ***complex load amplitude***.

$$\underline{\hat{\mathbf{v}}}(i\Omega) = \underline{\mathbf{K}^{-1}}(i\Omega) \cdot \underline{\hat{\mathbf{p}}}(i\Omega) = \underline{\mathbf{H}}(i\Omega) \cdot \underline{\hat{\mathbf{p}}}(i\Omega)$$

Matrix of Transfer Functions

A ***complex, frequency-dependent flexibility matrix H*** connects the ***complex load amplitude*** with the ***complex displacement amplitude***. The frequency-dependent flexibility matrix encompasses the ***transfer properties*** of the system. It transforms the absolute load amplitudes into the absolute displacement amplitudes and imparts a phase shift to each degree of freedom.

$$\underline{\hat{v}}(i\Omega) = \underline{\mathbf{H}}(i\Omega) \cdot \underline{\hat{p}}(i\Omega)$$

$$\underline{\mathbf{H}}(i\Omega) = \begin{bmatrix} \underline{H}_{11}(i\Omega) & \underline{H}_{12}(i\Omega) & \cdots & \underline{H}_{1N}(i\Omega) \\ \underline{H}_{21}(i\Omega) & \underline{H}_{22}(i\Omega) & \cdots & \underline{H}_{2N}(i\Omega) \\ \cdots & \cdots & \cdots & \cdots \\ \underline{H}_{N1}(i\Omega) & \underline{H}_{N2}(i\Omega) & \cdots & \underline{H}_{NN}(i\Omega) \end{bmatrix}$$

The matrix of transfer functions requires large storage space:

- **The stiffness, damping and mass matrices are sparsely populated matrices – the transfer matrix, however, is fully populated.**
- **Each element in H is a function of frequency, i.e. a data array of length N_{freg} (number of frequency steps).**

Special computing strategies are needed to find a compromise between storage requirement and computing speed.

Response in the Frequency Domain I

Now we assume arbitrary loading p(t):

$$\mathbf{M} \cdot \ddot{\mathbf{v}} + \mathbf{C} \cdot \dot{\mathbf{v}} + \mathbf{K} \cdot \mathbf{v} = \mathbf{p}$$

Any load can be expressed by a FOURIER transformation as an infinite sum of harmonic parts:

$$\mathbf{p}(t) \xrightarrow{\text{Fourier transformation}} \underline{\mathbf{P}}(i\Omega) = \begin{bmatrix} \underline{P}_1(i\Omega) \\ \underline{P}_2(i\Omega) \\ \cdots \\ \underline{P}_N(i\Omega) \end{bmatrix}$$

$$\mathbf{p}(t) = \frac{1}{2\pi} \int_{\Omega} \underline{\mathbf{P}}(i\Omega) e^{i\Omega t} d\Omega$$

$$\underline{\mathbf{P}}(i\Omega) = \int_{T} \mathbf{p}(t) e^{-i\Omega t} dt$$

The FOURIER transform of a vector is equal to a vector where each element is separately subjected to a FOURIER transformation.

Response in the Frequency Domain II

We substitute p(t) with the expression for the FOURIER synthesis. Now we have an infinite sum of harmonic parts on the right-hand side:

$$\mathbf{M}\cdot\ddot{\mathbf{v}}+\mathbf{C}\cdot\dot{\mathbf{v}}+\mathbf{K}\cdot\mathbf{v}=\frac{1}{2\pi}\int_{\Omega}\underline{\mathbf{P}}(i\Omega)e^{i\Omega t}d\Omega$$

The total displacement v(t) then can be expressed as the integral over individual solutions v_Ω. The solutions v_Ω due to harmonic excitation are known:

$$\mathbf{v}=\frac{1}{2\pi}\int_{\Omega}\mathbf{v}_\Omega(t)d\Omega$$

solution for harmonic excitation

$$\mathbf{v}_\Omega(t)=\hat{\underline{\mathbf{v}}}\cdot e^{i\Omega t}$$

$$\hat{\underline{\mathbf{v}}}=\underline{\mathbf{H}}(i\Omega)\cdot\hat{\underline{\mathbf{p}}}(i\Omega)=\underline{\mathbf{H}}(i\Omega)\cdot\underline{\mathbf{P}}(i\Omega)$$

$$\mathbf{v}=\frac{1}{2\pi}\int_{\Omega}\hat{\underline{\mathbf{v}}}\cdot e^{i\Omega t}d\Omega$$

inverse FOURIER transformation

$$\mathbf{v}(t)=\frac{1}{2\pi}\int_{\Omega}\underline{\mathbf{V}}(i\Omega)e^{i\Omega t}d\Omega$$

$$\underline{\mathbf{V}}(i\Omega)=\underline{\mathbf{H}}(i\Omega)\cdot\underline{\mathbf{P}}(i\Omega)$$

The FOURIER transform of the displacement vector is equal to the product of the transfer matrix with the FOURIER transform of the load vector.

Response in the Frequency Domain III

The FOURIER transform of the displacement vector is equal to the product of the transfer matrix with the FOURIER transform of the load vector.

$$\begin{bmatrix} V_1(i\Omega) \\ V_2(i\Omega) \\ \cdots \\ V_N(i\Omega) \end{bmatrix} = \begin{bmatrix} \underline{H}_{11}(i\Omega) & \underline{H}_{12}(i\Omega) & \cdots & \underline{H}_{1N}(i\Omega) \\ \underline{H}_{21}(i\Omega) & \underline{H}_{22}(i\Omega) & \cdots & \underline{H}_{2N}(i\Omega) \\ \cdots & \cdots & \cdots & \cdots \\ \underline{H}_{N1}(i\Omega) & \underline{H}_{N2}(i\Omega) & \cdots & \underline{H}_{NN}(i\Omega) \end{bmatrix} \begin{bmatrix} P_1(i\Omega) \\ P_2(i\Omega) \\ \cdots \\ P_N(i\Omega) \end{bmatrix}$$

An inverse FOURIER transformation would yield the time histories of the displacements:

$$\mathbf{V}(i\Omega) = \begin{bmatrix} \underline{V}_1(i\Omega) \\ \underline{V}_2(i\Omega) \\ \cdots \\ \underline{V}_N(i\Omega) \end{bmatrix} \xrightarrow{\text{Fourier synthesis}} \mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \cdots \\ v_N(t) \end{bmatrix} = \int_{\Omega} \mathbf{V}(i\Omega) e^{i\Omega t} d\Omega$$

Spectral Response I

The cross-spectrum between two displacement degrees of freedom k and j is defined by

$$\underline{S}_{v_k v_j}(i\Omega) = S_{vkj}(i\Omega) = \lim_{T/2\to\infty} \frac{1}{T} \underline{\tilde{V}}_k(i\Omega) \cdot \underline{V}_j(i\Omega)$$

The FOURIER transforms of the displacements k and j are computed by the product of the appropriate rows of the transfer matrix with the vector of load transforms:

$$\underline{V}_k = \begin{bmatrix} \underline{H}_{k1} & \underline{H}_{k2} & \cdots & \underline{H}_{kN} \end{bmatrix} \begin{bmatrix} \underline{P}_1 \\ \underline{P}_2 \\ \cdots \\ \underline{P}_N \end{bmatrix}$$

$$\underline{V}_j = \begin{bmatrix} \underline{H}_{j1} & \underline{H}_{j2} & \cdots & \underline{H}_{jN} \end{bmatrix} \begin{bmatrix} \underline{P}_1 \\ \underline{P}_2 \\ \cdots \\ \underline{P}_N \end{bmatrix} = \begin{bmatrix} \underline{P}_1 & \underline{P}_2 & \cdots & \underline{P}_N \end{bmatrix} \begin{bmatrix} \underline{H}_{j1} \\ \underline{H}_{j2} \\ \cdots \\ \underline{H}_{jN} \end{bmatrix}$$

$$\underline{\tilde{V}}_k \underline{V}_j = \begin{bmatrix} \underline{\tilde{H}}_{k1} & \underline{\tilde{H}}_{k2} & \cdots & \underline{\tilde{H}}_{kN} \end{bmatrix} \begin{bmatrix} \underline{\tilde{P}}_1 \\ \underline{\tilde{P}}_2 \\ \cdots \\ \underline{\tilde{P}}_N \end{bmatrix} \begin{bmatrix} \underline{P}_1 & \underline{P}_2 & \cdots & \underline{P}_N \end{bmatrix} \begin{bmatrix} \underline{H}_{j1} \\ \underline{H}_{j2} \\ \cdots \\ \underline{H}_{jN} \end{bmatrix}$$

$$\underline{\tilde{V}}_k \underline{V}_j = \begin{bmatrix} \underline{\tilde{H}}_{k1} & \underline{\tilde{H}}_{k2} & \cdots & \underline{\tilde{H}}_{kN} \end{bmatrix} \begin{bmatrix} \underline{\tilde{P}}_1 \underline{P}_1 & \underline{\tilde{P}}_1 \underline{P}_2 & \cdots & \underline{\tilde{P}}_1 \underline{P}_N \\ \underline{\tilde{P}}_2 \underline{P}_1 & \underline{\tilde{P}}_2 \underline{P}_2 & \cdots & \underline{\tilde{P}}_2 \underline{P}_N \\ \cdots & \cdots & \cdots & \cdots \\ \underline{\tilde{P}}_N \underline{P}_1 & \underline{\tilde{P}}_N \underline{P}_2 & \cdots & \underline{\tilde{P}}_N \underline{P}_N \end{bmatrix} \begin{bmatrix} \underline{H}_{j1} \\ \underline{H}_{j2} \\ \cdots \\ \underline{H}_{jN} \end{bmatrix}$$

Spectral Response II

The matrix element S_{vkj} of the spectral matrix S_v of the displacements is then computed as the product of the k^{th} row of the transfer matrix H with the spectral matrix S_p of the loads with the j^{th} column of the conjugate complex of H.

$$S_{vkj} = \lim_{T/2\to\infty} \frac{1}{T} \underline{\tilde{V}}_k \underline{V}_j = \begin{bmatrix} \underline{\tilde{H}}_{k1} & \underline{\tilde{H}}_{k2} & \cdots & \underline{\tilde{H}}_{kN} \end{bmatrix} \begin{bmatrix} \underline{S}_{p11} & \underline{S}_{p12} & \cdots & \underline{S}_{p1N} \\ \underline{S}_{p21} & \underline{S}_{p22} & \cdots & \underline{S}_{p2N} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{S}_{pN1} & \underline{S}_{pN2} & \cdots & \underline{S}_{pNN} \end{bmatrix} \begin{bmatrix} \underline{H}_{j1} \\ \underline{H}_{j2} \\ \cdots \\ \underline{H}_{jN} \end{bmatrix}$$

$$S_{pkj} = \lim_{T/2\to\infty} \frac{1}{T} \underline{\tilde{P}}_k \underline{P}_j$$

We find the entire spectral matrix S_v by extending the product over all rows and columns of H:

$$\begin{bmatrix} \underline{S}_{v11} & \underline{S}_{v12} & \cdots & \underline{S}_{v1N} \\ \underline{S}_{v21} & \underline{S}_{v22} & \cdots & \underline{S}_{v2N} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{S}_{vN1} & \underline{S}_{vN2} & \cdots & \underline{S}_{vNN} \end{bmatrix} = \begin{bmatrix} \underline{\tilde{H}}_{11} & \underline{\tilde{H}}_{12} & \cdots & \underline{\tilde{H}}_{1N} \\ \underline{\tilde{H}}_{21} & \underline{H}_{22} & \cdots & \underline{\tilde{H}}_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{\tilde{H}}_{N1} & \underline{\tilde{H}}_{N2} & \cdots & \underline{H}_{NN} \end{bmatrix} \begin{bmatrix} \underline{S}_{p11} & \underline{S}_{p12} & \cdots & \underline{S}_{p1N} \\ \underline{S}_{p21} & \underline{S}_{p22} & \cdots & \underline{S}_{p2N} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{S}_{pN1} & \underline{S}_{pN2} & \cdots & \underline{S}_{pNN} \end{bmatrix} \begin{bmatrix} \underline{H}_{11} & \underline{H}_{12} & \cdots & \underline{H}_{1N} \\ \underline{H}_{21} & \underline{H}_{22} & \cdots & \underline{H}_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \underline{H}_{N1} & \underline{H}_{N2} & \cdots & \underline{H}_{NN} \end{bmatrix}$$

$$\mathbf{S}_v = \tilde{\mathbf{H}} \cdot \mathbf{S}_p \cdot \mathbf{H}$$

Derived Variables

The transfer matrix connects the spectral matrix of the deformations (displacements and rotations) with the spectral matrix of the loads. From the auto-spectra of the deformations we can compute their standard deviations and from the standard deviations plus a suitable peak g factor the maximum expected values.

$$\sigma_{v_k}^2 = \int_{\Omega} S_{v_k}(\Omega)\,d\Omega \quad \Rightarrow \quad v_{k,max} = \mu_{vk} + g \cdot \sigma_{vk}$$

We need, however, further results which can be derived from the deformations:

- **Velocities and accelerations**
- **Internal force variables:**
  - **stress resultants such as N, Q, M,**
  - **stresses such as normal stresses σ or shear stresses τ.**

FOURIER Transforms of Velocities & Accelerations

Inverse FOURIER transformations:

$$v(t)=\frac{1}{2\pi}\int_{\Omega}\underline{V}(i\Omega)e^{i\Omega t}d\Omega \qquad \dot{v}(t)=\frac{1}{2\pi}\int_{\Omega}\dot{\underline{V}}(i\Omega)e^{i\Omega t}d\Omega \qquad \ddot{v}(t)=\frac{1}{2\pi}\int_{\Omega}\ddot{\underline{V}}(i\Omega)e^{i\Omega t}d\Omega$$

The velocity is the time derivative of the displacement:

$$\dot{v}(t)=\frac{d}{d}=\frac{1}{2\pi}\frac{d}{d}\left[\int_{\Omega}\underline{V}(i\Omega)e^{i\Omega t}d\Omega\right]=\frac{1}{2\pi}\int_{\Omega}i\Omega\underline{V}(i\Omega)e^{i\Omega t}d\Omega$$

$$\dot{\underline{V}}=i\Omega\underline{V}(i\Omega) \quad \Rightarrow \quad \ddot{\underline{V}}=i^2\Omega^2\underline{V}(i\Omega)$$

We can gain the FOURIER transforms of the velocities and accelerations directly from the FOURIER transforms of the deformations.

Spectra of Velocities & Accelerations

FOURIER transforms of two degrees of freedom k and j:

$$\underline{V}_k = a + b\,i \;\Rightarrow\; \dot{\underline{V}}_k = i\Omega\underline{V}_k = \Omega(a - ib)$$

$$\underline{V}_j = c + d\,i \;\Rightarrow\; \dot{\underline{V}}_j = i\Omega\underline{V}_j = \Omega(c - id)$$

For the spectrum of the displacement we need:

$$\tilde{\underline{V}}_k \cdot \underline{V}_j = (a - bi)(c + di) = (ac + bd) + i(ad - bc)$$

For the spectrum of the velocity we need:

$$\tilde{\dot{\underline{V}}}_k \dot{\underline{V}}_j \cdot = \Omega(-ai - b)\Omega(ci - d) = \Omega^2[(ac + bd) + i(ad - bc)] = \Omega^2 \cdot \tilde{\underline{V}}_k \cdot \underline{V}_j$$

$$\Downarrow$$

$$\mathbf{S}_{\dot{v}} = \Omega^2 \mathbf{S}_v \;\Rightarrow\; \mathbf{S}_{\ddot{v}} = \Omega^4 \mathbf{S}_v \;\Rightarrow\; \sigma^2_{\ddot{v}} = \int_{\Omega} \Omega^4 \mathbf{S}_v \, d\Omega \neq \Omega^4 \cdot \sigma^2_v$$

Derived Variables II: Inner Forces

The stress resultants can be calculated from the nodal degrees of freedom of the element:

general case

$$\mathbf{s}_e = \mathbf{B}_e \cdot \mathbf{v}_e$$

beam element

$$\mathbf{s}_e = \mathbf{k}_e \cdot \mathbf{v}_e$$

The nodal degrees of the element are given by the spectral matrix S_{ve}, containing both auto- and cross-spectra. These can be extracted from the spectral matrix S_v of the system. Since the expressions for the calculation of s_e are linear, we can simply write:

general case

$$\mathbf{S}_{se}(i\Omega) = \mathbf{B}_e \cdot \mathbf{S}_{ve} \cdot \mathbf{B}_e^T$$

beam element

$$\mathbf{S}_{se}(i\Omega) = \mathbf{k}_e \cdot \mathbf{S}_{ve} \cdot \mathbf{k}_e^T$$

We get the full spectral information on the element stress resultants: auto- and cross-spectral properties. From the auto-spectra we calculate the standard deviations, and from them the maximum expected values.

Derived Variables III: Stresses

We compute the stresses from the stress resultants, e.g. the normal stress σ from the normal force N and the two bending moment M_y and M_z:

$$\sigma(y,z) = \frac{N}{A} + \frac{M_y}{I_{yy}} z - \frac{M_z}{I_{zz}} y \quad \Rightarrow \quad \sigma(y,z) = \begin{bmatrix} \frac{1}{A} & \frac{z}{I_{yy}} & -\frac{y}{I_{zz}} \end{bmatrix} \begin{bmatrix} N \\ M_y \\ M_z \end{bmatrix}$$

It would be wrong to take the maximum expected values for N, M_y, M_z and use them to find the maximum expected value for σ, since N_{max}, M_{ymax} and M_{zmax} do not occur at the same time instance. We have to take into account their correlation which can be extracted from the spectral matrix S_{se}. Again we have a linear relationship between σ and the stress resultants, so we can obtain the auto-spectrum of σ by:

$$S_\sigma = \begin{bmatrix} \frac{1}{A} & \frac{z}{I_{yy}} & -\frac{y}{I_{zz}} \end{bmatrix} \begin{bmatrix} S_{NN} & S_{NM_y} & S_{NM_z} \\ S_{M_yN} & S_{M_yM_y} & S_{M_yM_z} \\ S_{M_zN} & S_{M_zM_y} & S_{M_zM_z} \end{bmatrix} \begin{bmatrix} 1/A \\ z/I_{yy} \\ -y/I_{zz} \end{bmatrix}$$

Application II:

Academic Test : Cantilever Beam with 4 Correlated Loads

Introduction

We have developed a new algorithm to compute the structural response for stochastic excitation. Structural response here is equivalent to the standard deviation of any desired mechanical variable. Now we need to check whether our theoretical deliberations and our implementaion are correct or not. Such a check is usually performed by checking against some reference solution whose correctness is not in doubt. For example:

Example 1: Duhamel Integral

The Duhamel integral is an analytical solution of an SDOF oscillator under arbitrary loads with homogeneous initial conditions. The numerical solution of the integral introduces a numerical error which must remain within acceptable bounds. The algorithm could be easily checked by comparing Duhamel solutions with different time steps to analytical solutions which could be found by solving the linear differential equations. So here the check was done against a mathematically correct solution.

Example 2: Frequency Domain Calculation

The frequency domain solution uses the FFT algorithm, i.e. it uses discrete data. The generation of the discrete data depends on the length T of the time series and the size of the time step Δt. Again we can check the accuracy by a comparison with analytical solutions or, alternatively, by a comparison with Duhamel solutions whose correctness have been proven before.

Example 3: Modal Superposition

The correctness of the modal superposition algorithm could be checked against a reference solution computed by direct time integration for a sufficiently small time step. The results must (and did) become identical if all mode shapes are taken into account and if modal damping is set to the damping measures calculated from the Rayleigh damping matrix.

Scope of the Check

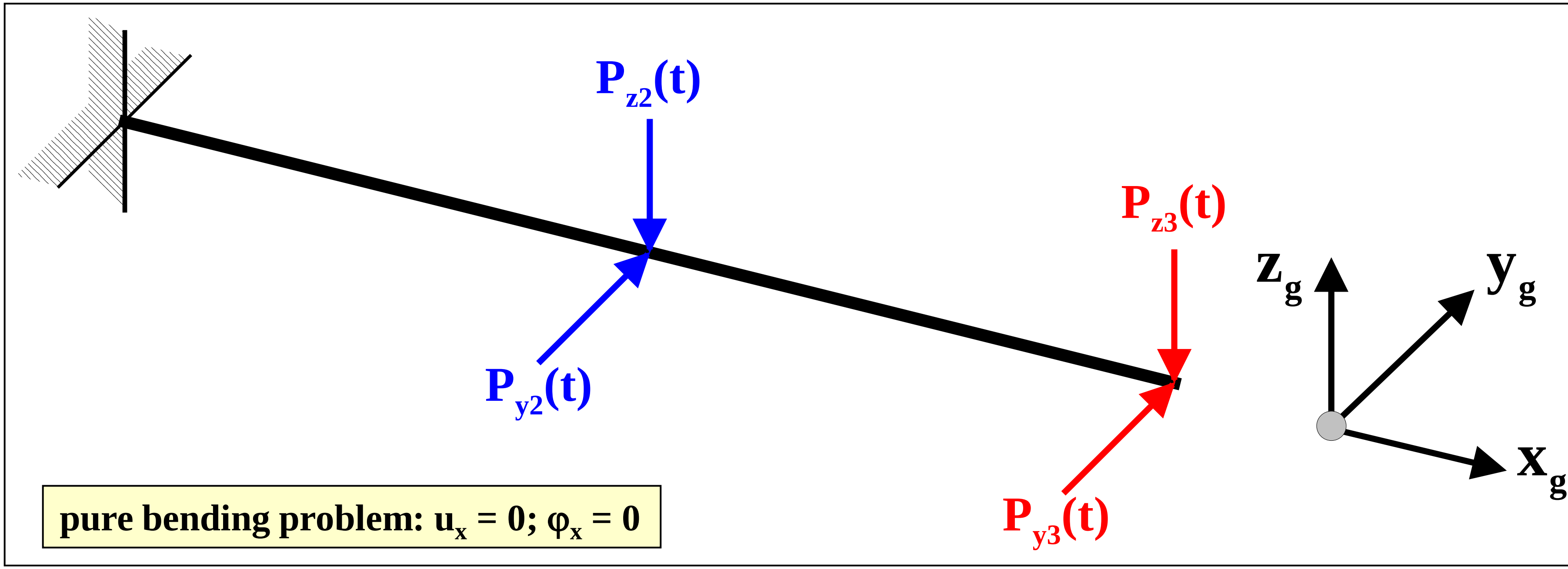
Here we have the problem where a reference solution does not exist. There is no alternative manner to describe stochastic processes: the spectral approach is the only possible way.

A time domain approach requires time histories of the excitation which simply do not exist for a stochastic process. It is only possible to generate by some suitable algorithm numerical samples of the underlying stochastic process. These are only approximations: both the generation algorithm and the finite length of the sample introduce numerical errors in the time domain solution.

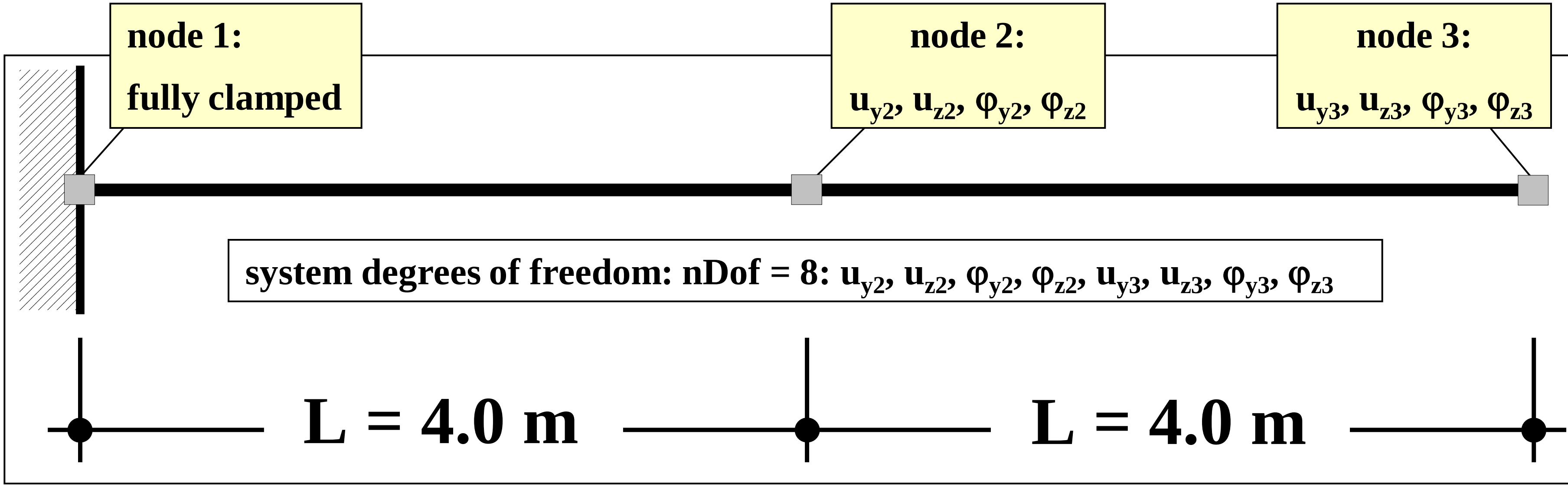
So we cannot check the spectral approach against a ***true solution***, we can only compare two ***equivalent approaches*** which should produce, instead of ***identical results***, results which are only ***similar***. Deviations between the two results are unavoidable, but should remain within statistically significant bounds. The check proceeds as follows:

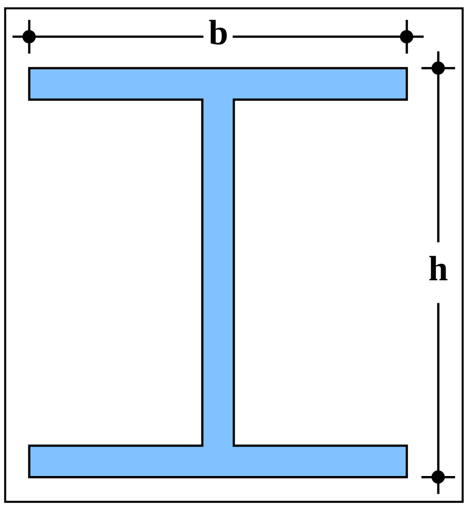
1. A ***smooth analytical auto-spectrum*** is chosen (c.f. example with 2 correlated load processes) for the load.
2. A single time history of the load is generated by the WAWS-algorithm for this target spectrum. The time history must have sufficient length to be statistically valid.
3. Further time histories are produced from the generated one by ***shifting the time axis***. A time domain analysis is performed for theses histories. Standard deviations for displacements, bending moments, and stresses are computed from the structural time domain response.
4. The ***cross-spectral matrix*** is computed from the ***actually used time histories of the loads***. The auto-spectra are smooth, but the cross-spectra show the typical strong fluctuations. A spectral analysis is done for this spectral loading. Standard deviations are computed by integrating the spectra of the response.
5. The computed standard deviations are compared.

Example:
Cantilever Beam with 4 Correlated Loads



Cantilever Beam: Discretization





Cross section: HEA 300

- I_{yy} = 18263 cm^4
- I_{zz} = 6310 cm^4
- h = 290 mm
- b = 300 mm

Material:

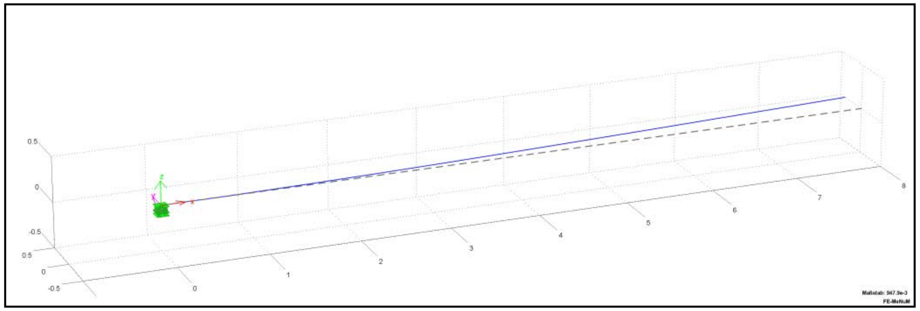
- steel: $E = 2.1 \cdot 10^8$ kN/m^2

Mass:

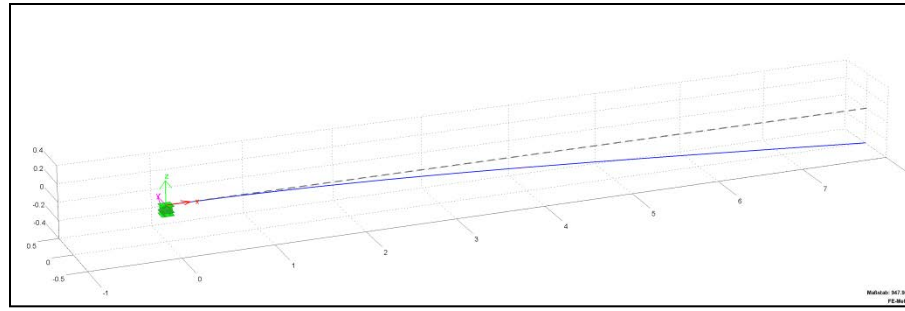
- m = 2.81 to/m

Eigenfrequency Analysis

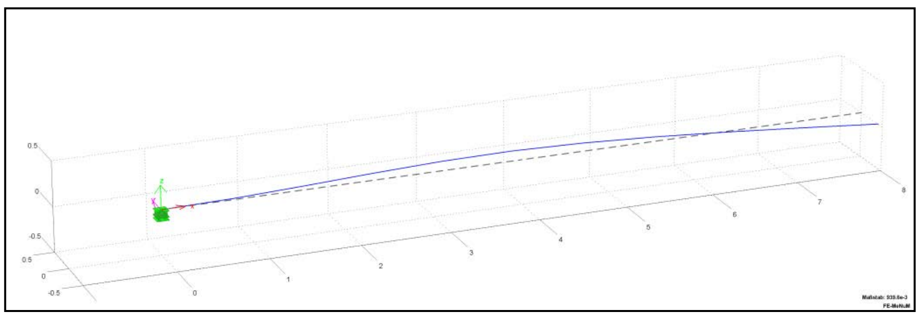
mode 1 – horizontal bending – f_1 = 0.60 Hz



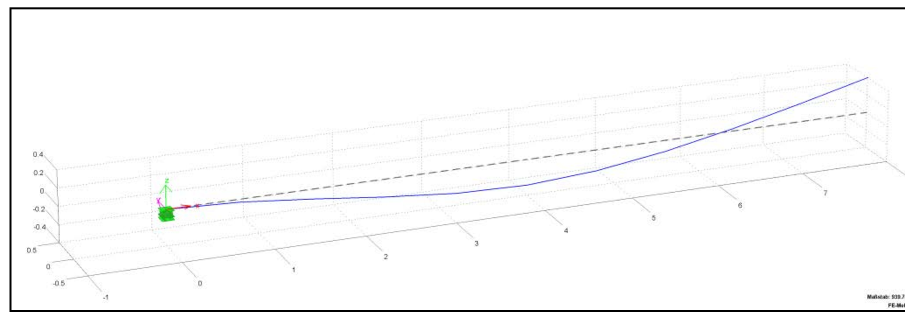
mode 2 – vertical bending – f_2 = 1.02 Hz



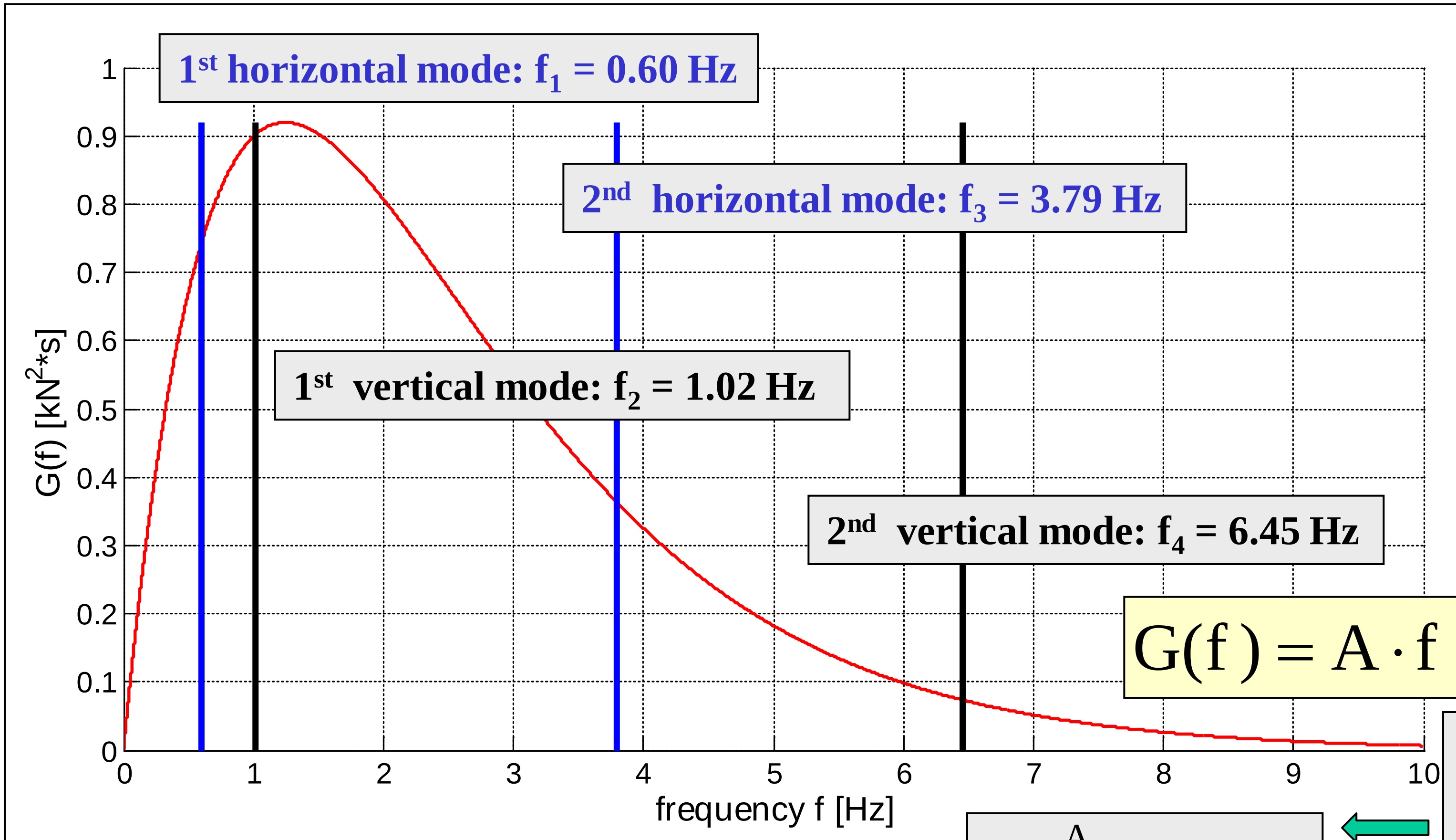
mode 3 – horizontal bending – f_3 = 3.79 Hz



mode 4 – vertical bending – f_4 = 6.45 Hz



Stochastic Loading: Auto-Spectrum



$$G(f) = A \cdot f \cdot e^{-B \cdot f}$$

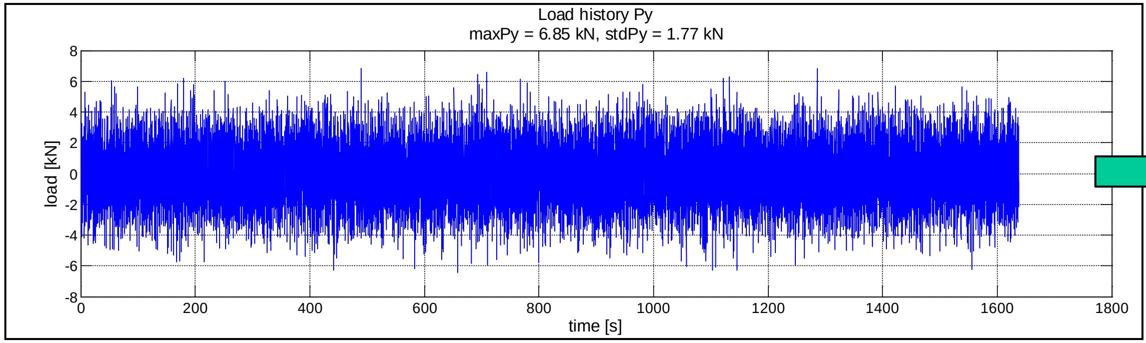
A = 2.0

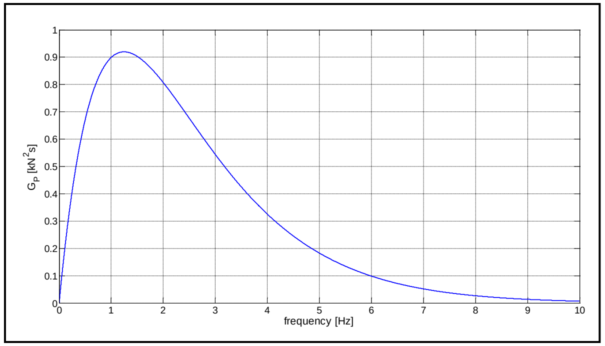
B = 0.8

$$\sigma^2 = \frac{A}{B^2} = 3.1250\,\text{kN}^2$$

Stochastic Loading: Spectral Properties

A load history has been generated by the WAWS algorithm, using nFreq=16382=2^{14} frequencies. The resulting time history has nStep=2*nFreq time steps with an Δt_{gen}=0.05 s. This covers the frequency range up until f_{max}=10.0 Hz (Nyquist frequency). The auto-spectrum from the time history possesses the smooth curve shown below.





An "exact" numerical integration of the spectrum from f = 0 Hz to $f_{nyquist}$ = 10.0 Hz yields the numerical standard deviation $\sigma_{spec,num}$ = 1.7652 kN instead of the true standard deviation σ_{true} = 1.7678 kN. Cutting off the frequency axis at $f_{nyquist}$ neglects the contributions of the frequencies f > $f_{nyquist}$ which introduces a numerical error ε in the load spectrum of ε = 0.15 %.

$$\sigma_{true} = \sqrt{\frac{A}{B^2}} = 1.7678\,\text{kN}$$

$$\sigma_{spec,num} = \int_0^{10\,\text{Hz}} G(f)\,df = 1.7651\,\text{kN}$$

$$\varepsilon = 100\,\frac{\sigma_{spec,num} - \sigma_{true}}{\sigma_{true}} = -0.15\%$$

Stochastic Loading: Correlation

The other three load histories are created by cutting off [2 4 6] time steps from the beginning of the first history and pasting them onto the tail of the history. This is equivalent to a shift of the time axis which, as we have seen before, does not change the auto-spectrum. So all auto-spectra are identical. The time shift, however, leads to a decorrelation which would be visible in the randomness of the phase angles of the cross-spectra. The *overall correlation* can be seen in the matrix of "*cross-variances*" C_p which is computed by integrating the cross-spectral matrix over the entire frequency range. Normalizing with respect to the diagonal elements gives C_{pn}. The correlation between the variances of the time histories lies in the range of 7.5 % to 14.5 %.

$$\mathbf{C}_p = \int_0^{f_{max}} \mathbf{G}_p df = \begin{bmatrix} 3.1156 & 0.4527 & -0.3837 & -0.2353 \\ 0.4527 & 3.1156 & 0.4527 & -0.3325 \\ -0.3837 & 0.4527 & 3.1156 & -0.3837 \\ -0.2353 & -0.3325 & -0.3837 & 3.1156 \end{bmatrix}$$

$$\mathbf{C}_{pn} = \begin{bmatrix} 1.0000 & 0.1453 & -0.1232 & -0.0755 \\ 0.1453 & 1.0000 & 0.1453 & -0.1067 \\ -0.1232 & 0.1453 & 1.0000 & -0.1232 \\ -0.0755 & -0.1067 & -0.1232 & 1.0000 \end{bmatrix}$$

Time Domain Analysis: Properties

The standard deviation σ_{th} of the generated time history has been computed from the 2^{15} values of the dicrete data to σ_{th} = 1.7651 kN which is, within the 5 significant decimals, exactly the value of the target spectrum in the frequency range 0 Hz ≤ f ≤ 10 Hz. The time series is therefore sufficiently long to reduce the numerical error for the standard deviation to practically zero.

The load spectrum is virtually zero for frequencies above 10 Hz. Only the first 4 eigenmodes lie in the relevant spectral range (f_5=12.8 Hz is already above 10 Hz). So the highest eigenmode that needs to be calculated with sufficiently small time steps is mode 4. The analysis time step Δt_{FE} is determined by mode #4.

$$f_4 = 6.45\,\text{Hz} \;\rightarrow\; T_4 = \frac{1}{f_4} = 0.155\,\text{s} \;\rightarrow\; \frac{T_4}{\Delta t_{gen}} = \frac{0.155\,\text{s}}{0.05\,\text{s}} = 3.1$$

The time increment Δt_{gen} is too coarse for the direct time integration. Each increment Δt_{gen} has been subdivided into 10 sub-increments, so that $\Delta t_{FE} = \Delta t_{gen}/10$. Damping has been defined with regard to the horizontal bending modes, i.e. modes 1 and 3.

Calculation properties in time domain:

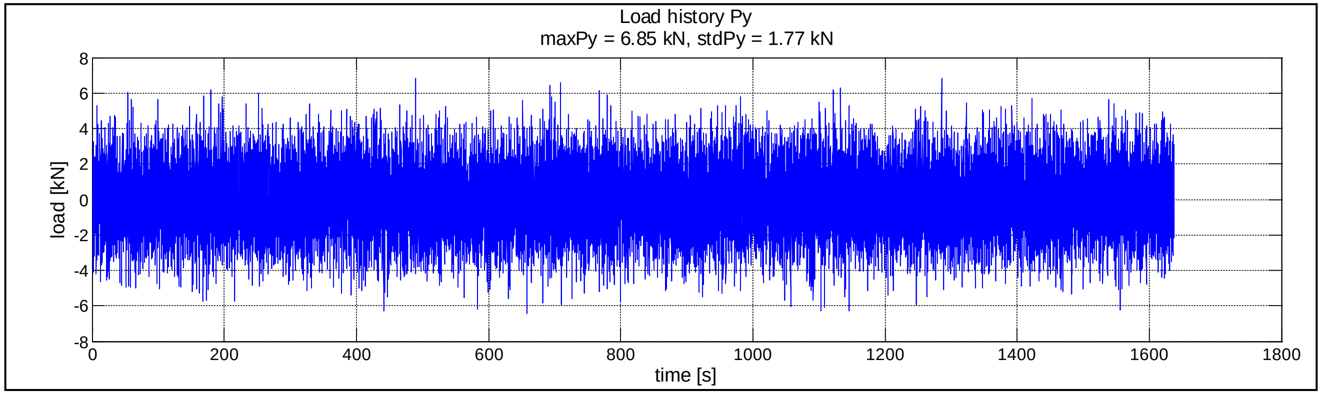
- T_{max} = 1638.3 s
- Δt_{FE} = 0.005 s
- algorithm: CAA

Damping properties:

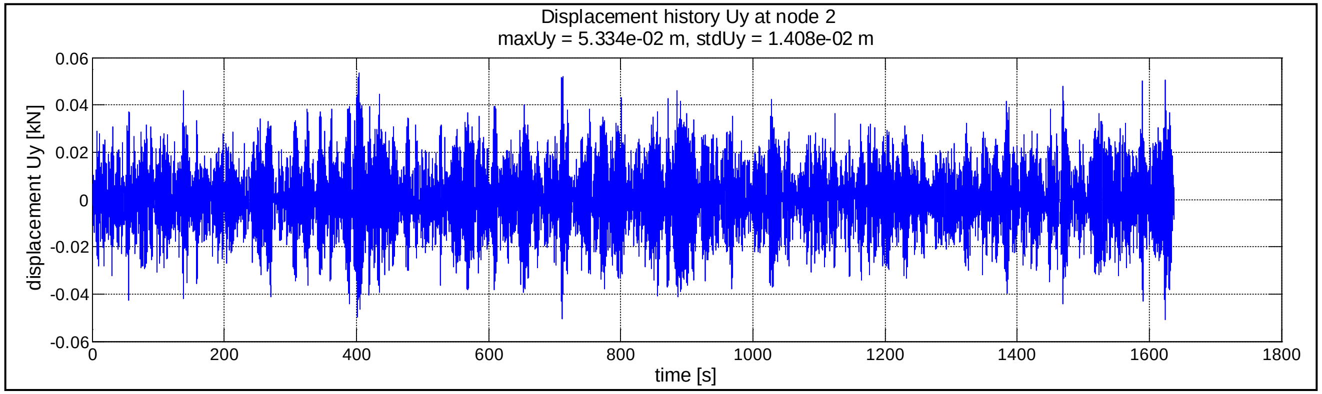
- mode 1: ξ= 5%
- mode 3: ξ= 5%

TD Simulation via Direct Time Integration: Load Histories ⇒ Displacement Histories

Given: 4 load histories: $P_{y2}(t)$, $P_{z2}(t)$, $P_{y3}(t)$, $P_{z3}(t)$

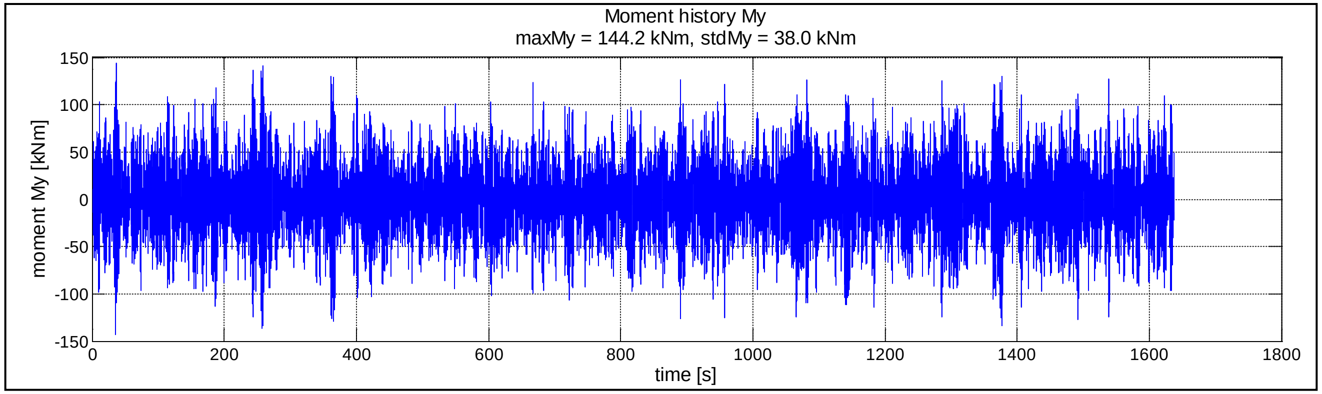


Computed: 8 deformation histories: $U_{y2}(t)$, $U_{z2}(t)$, $\Phi_{y2}(t)$, $\Phi_{z2}(t)$, $U_{y3}(t)$, $U_{z3}(t)$), $\Phi_{y3}(t)$, $\Phi_{z3}(t)$

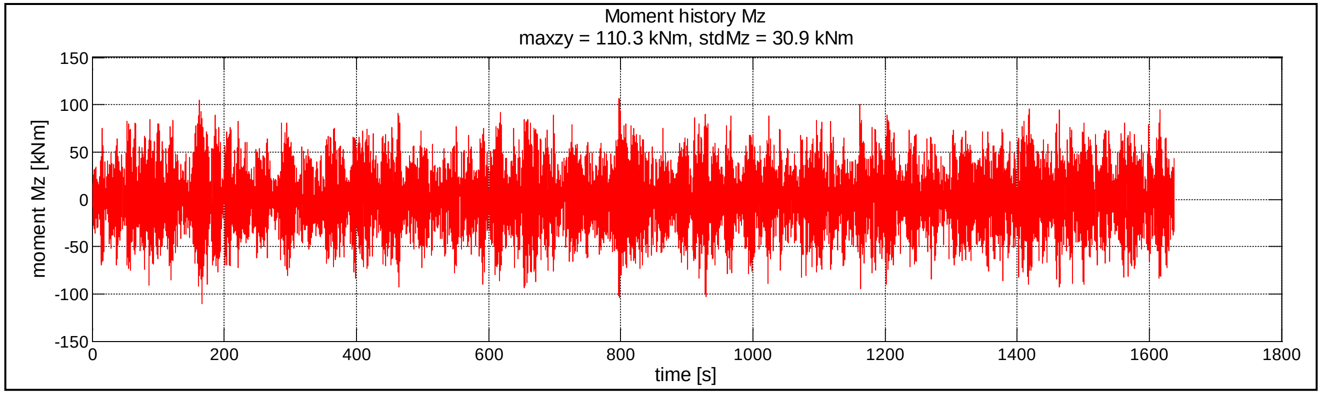


Post Processing I:
Displacement Histories ⇒ Moment Histories

Moment history $M_y(t)$ element 1, left node (clamped support)

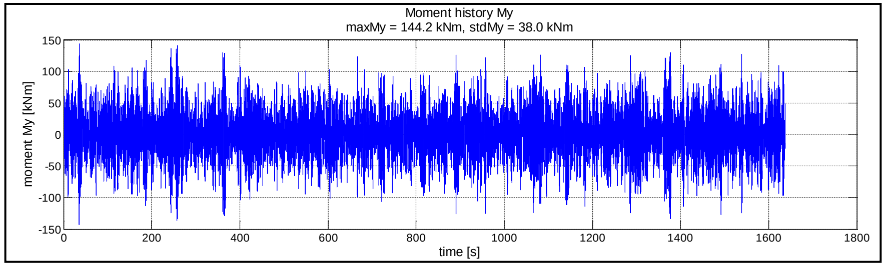


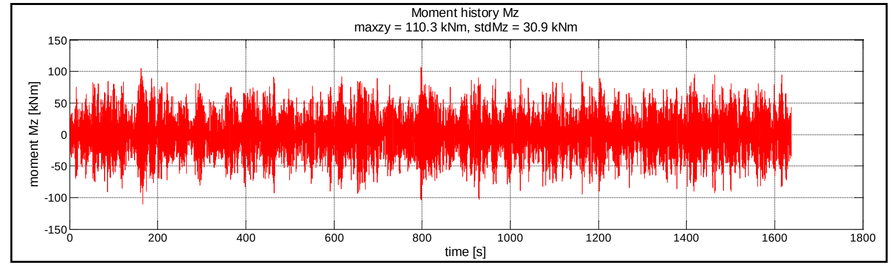
Moment history $M_z(t)$ element 1, left node (clamped support)



Post Processing II: Moment Histories ⇒ Stress Histories

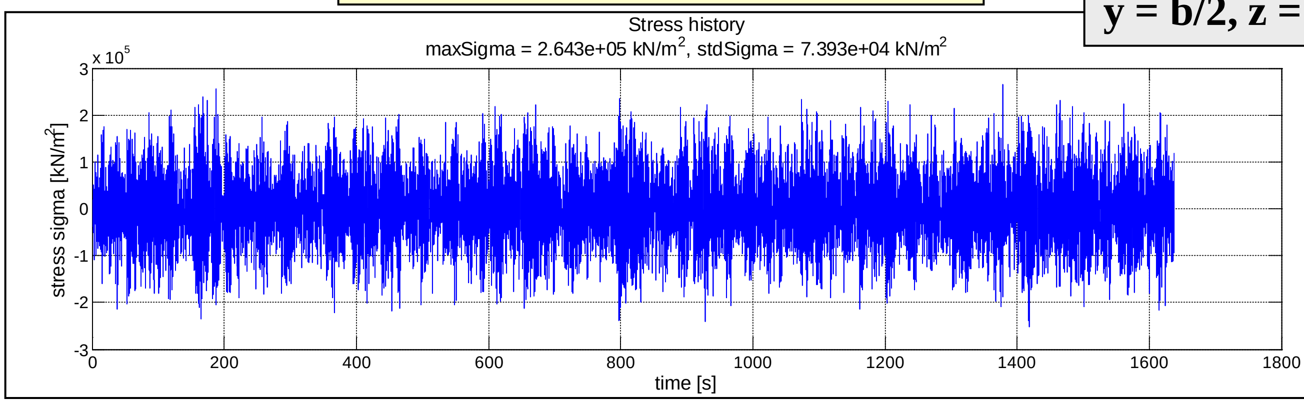
Moment histories $M_y(t)$ and $M_z(t)$





$$\sigma(y,z,t) = \frac{N(t)}{A} + \frac{M_y(t)}{I_{yy}} z - \frac{M_z(t)}{I_{zz}} y$$

y = b/2, z = h/2



TD Simulation: Scatter of the Results

Each time history contains, due to its limited length, only partial information regarding the underlying stochastic process. Therefore we must perform not one single analysis, but a whole ***batch of simulations***. The statistics we extract from the simulations show a certain scatter. Relevant parameters are:

1. The maximum values. EC 8, Part 2 (earthquakes: bridges), e.g. states (4.2.4.3 Bemessungs-Antwortgrößen): "*Wird eine nicht-lineare, dynamische Berechnung für mindestens sieben unabhängige Paare horizontaler Bodenbewegungen durchgeführt, so kann der Mittelwert der jeweiligen Antworten als Bemessungswert der Zustandsgrößen verwendet werden, es sei denn, es werden in diesem Teil andere Anforderungen gestellt. Wenn weniger als sieben nicht-lineare dynamische Berechnungen für die entsprechenden unabhängigen Paare von Eingabezeitverläufen durchgeführt werden, müssen die Größtwerte der Ergebnisse als Bemessungszustandsgrößen verwendet werden.*"
2. The standard deviation as the basis of a design with peak factors.

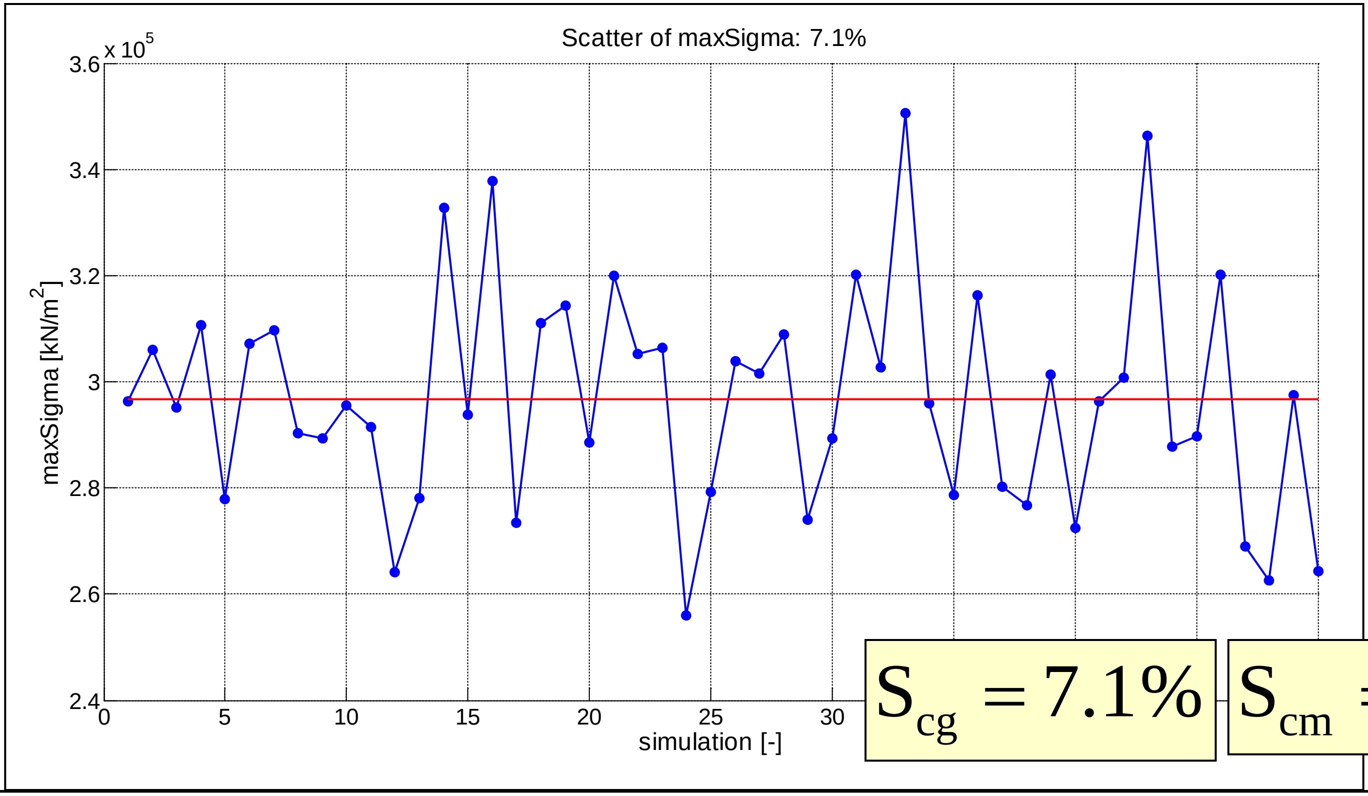
For the scatter we can define two scatter parameters:

1. ***Global scatter*** S_{cg}, defined as the ratio of the standard deviation of the statistic value to its mean value.
2. ***Maximum scatter*** S_{cm}, defined as the ratio of the maximum deviation to the mean value.

The scatter depends on the type of the statistic variable.

Scatter of the Maximum Value of the Stress σ_{xx}

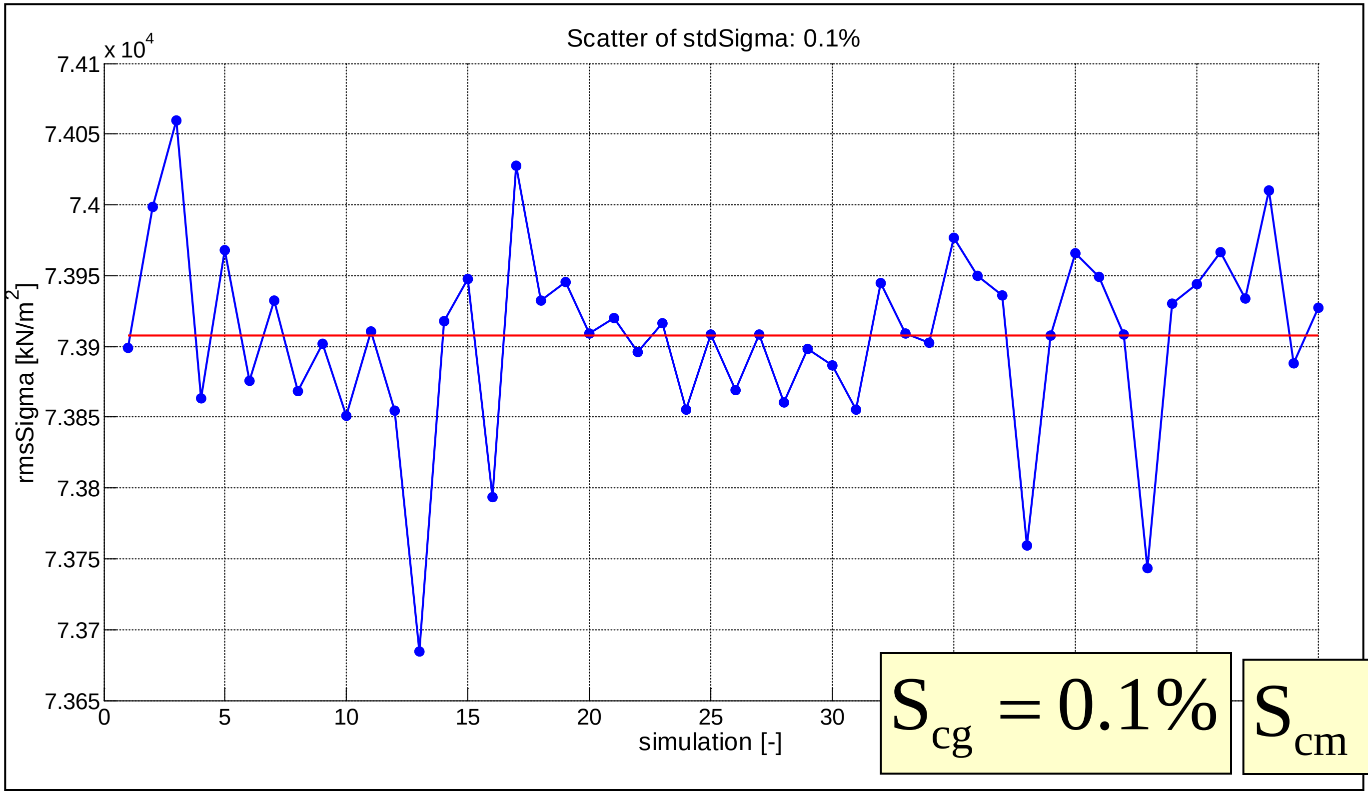
There have been performed 50 simulations, each consisting of: generation of the load histories, time-domain analysis, and post-processing. The maximum stresses resulting from this large data dump are plotted below. We observe an overall scatter of 7 %; the maximum scatter, however, can go up to 18 %! Obviously a rather large number of simulations is necessary to obtain statistically reliable values for the maximum stress. In wind engineering a typical number is 30 time histories.



$$S_{cg} = 7.1\% \qquad S_{cm} = 18.1\%$$

Scatter of the Standard Deviation of the Stress

The standard deviation is a much more stable variable. The overall scatter drops to 0.1%. The maximum scatter is again larger than the overall scatter, but it is also of a negligible order. So even one simulation would be sufficient to obtain a very good estimate of the true standard deviation. We would now, however, need a suitably defined peak factor to compute a statistically reliable value of the maximum stress.



$S_{cg} = 0.1\%$

$S_{cm} = 0.3\%$

Spectral Analysis: Spectral Matrix of the Loads

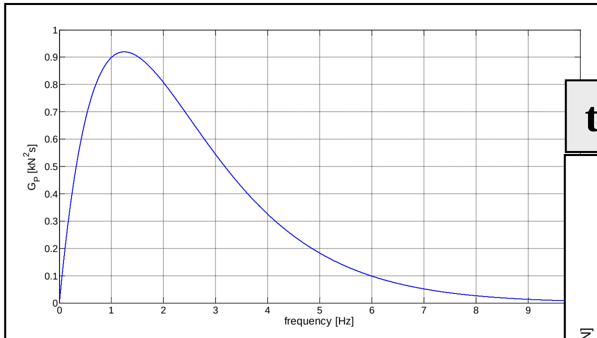
The four load histories lead to a cross-spectral matrix of dimension 4 x 4 where each matrix element is an array of nFreq=16382 values (the values are real for the auto-spectra and complex for the cross-spectra). The cross-spectra of the loads are stored in the nDof x nDof (nDof=8) cross-spectral matrix of the system loads. All entries connected to the rotational degrees of freedom are empty.

spectral matrix of the loads G_p

$$
G_p = \left[\begin{array}{cccccccccc}
 & u_{y2} & u_{z2} & \varphi_{y2} & \varphi_{z2} & u_{y3} & u_{z3} & \varphi_{y3} & \varphi_{z3} & \\
u_{y2} & G_{y2,y2} & G_{y2,z2} & [\,] & [\,] & G_{y2,y3} & G_{y2,z3} & [\,] & [\,] & u_{y2} \\
u_{z2} & G_{z2,y2} & G_{z2,z2} & [\,] & [\,] & G_{z2,y3} & G_{z2,z3} & [\,] & [\,] & u_{z2} \\
\varphi_{y2} & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & \varphi_{y2} \\
\varphi_{z2} & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & \varphi_{z2} \\
u_{y3} & G_{y3,y2} & G_{y3,z2} & [\,] & [\,] & G_{y3,y3} & G_{y3,z3} & [\,] & [\,] & u_{y3} \\
u_{z3} & G_{z3,y2} & G_{z3,z3} & [\,] & [\,] & G_{z3,y3} & G_{z3,z3} & [\,] & [\,] & u_{z3} \\
\varphi_{y3} & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & \varphi_{y3} \\
\varphi_{z3} & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & [\,] & \varphi_{z3} \\
 & u_{y2} & u_{z2} & \varphi_{y2} & \varphi_{z2} & u_{y3} & u_{z3} & \varphi_{y3} & \varphi_{z3} &
\end{array}\right]
$$

Spectral Analysis: $G_p \Rightarrow \chi_{up} \Rightarrow G_u$

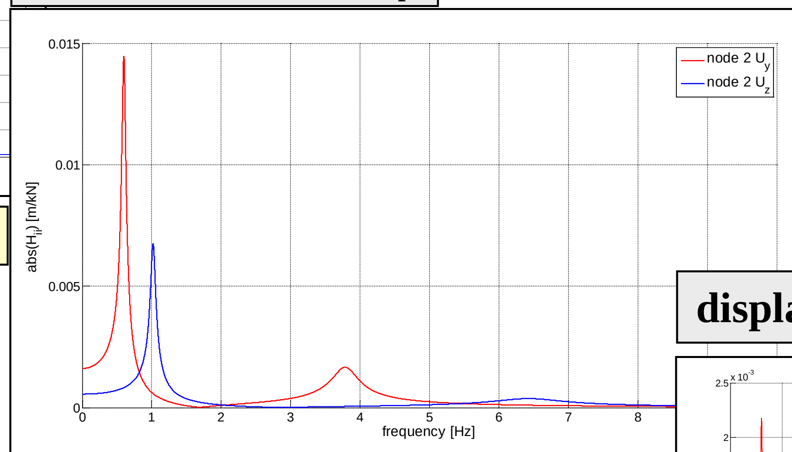
load spectrum G_p



4 x 4 non-zero entries of length nFreq

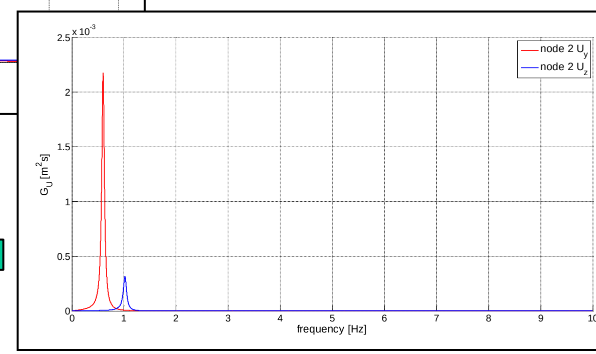
transfer function H_{up}

8 x 8 non-zero entries of length nFreq



$$G_u = \tilde{H} \cdot G_p \cdot H$$

displacement spectrum G_u

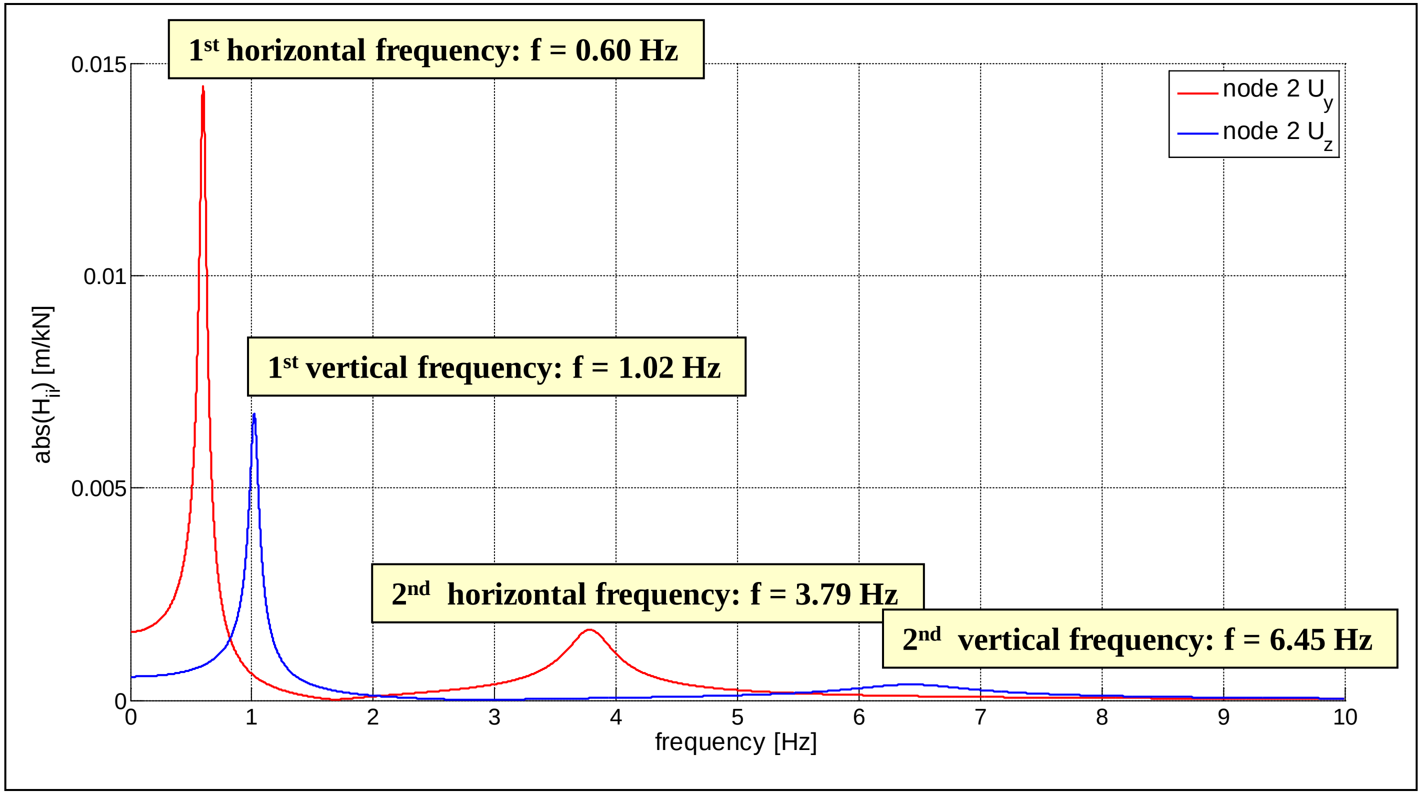


8 x 8 non-zero entries of length nFreq

$$\sigma_u^2 = \int_0^{f_{max}} G_u(f)\,df$$

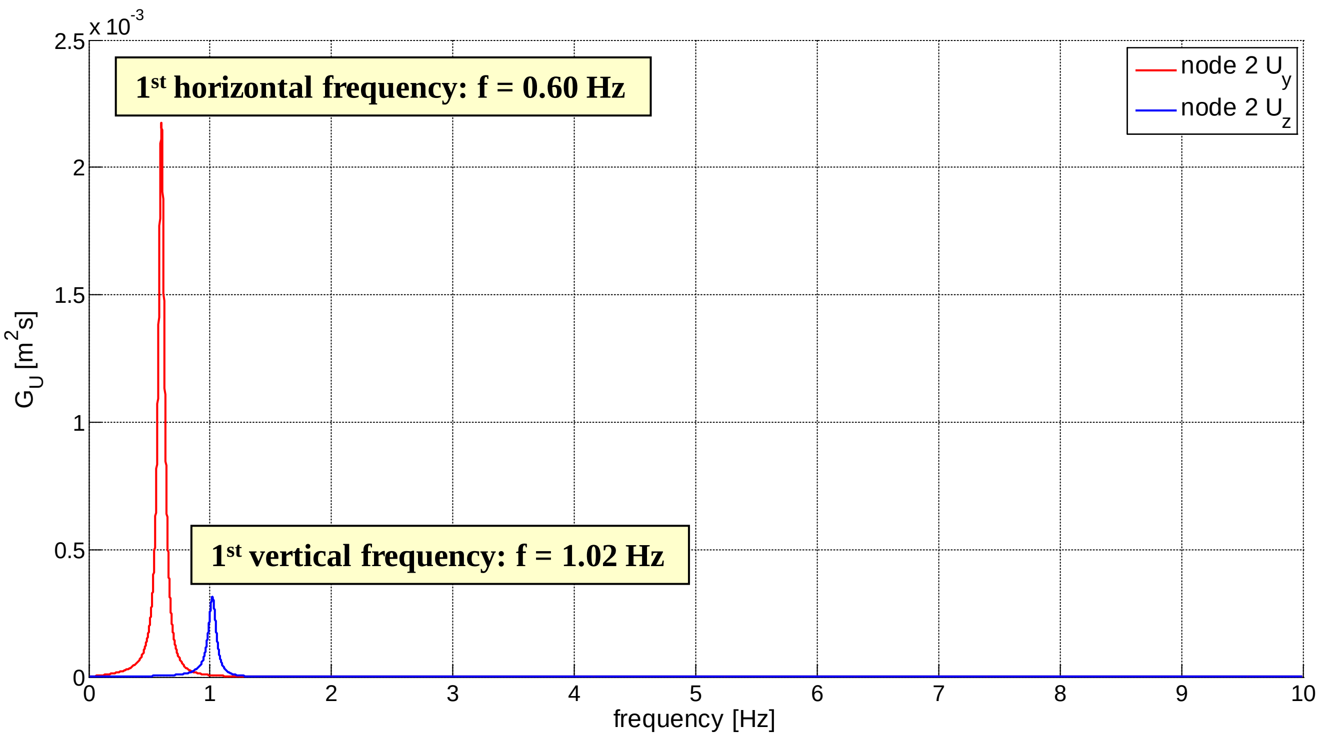
Diagonal Entries in H_{up}: $p_{y2} \Rightarrow u_{y2}$ and $p_{z2} \Rightarrow u_{z2}$

The first four eigenfrequencies can be seen in the transfer functions. The dynamic amplification of the 1st modes in the two bending directions dominate – the 2nd modes have amplifications of less than 20 % than the 1st ones.



Auto-Spectra of the Displacements u_{y2} and u_{z2}

The first modes dominate the response of the structure.



Spectral Matrix of the Stress Resultants

The element stiffness matrix of the spatial beam element possesses 12 element degrees of freedom. For element 1 only 4 of them are active: u_{y2}, u_{z2}, φ_{y2}, φ_{z2}. The corresponding 4 x 4 matrix containing the cross-spectra for these degrees of freedom are extracted from the system matrix G_u and copied to the appropriate places of the element matrix G_{ue}.

$$\mathbf{G}_{ue} = \begin{bmatrix}
[] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] \\
[] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] \\
[] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] \\
[] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] \\
[] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] \\
[] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] \\
[] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] \\
[] & [] & [] & [] & [] & [] & [] & G_{u11} & G_{u12} & [] & G_{u12} & G_{u14} \\
[] & [] & [] & [] & [] & [] & [] & G_{u21} & G_{u22} & [] & G_{u23} & G_{u24} \\
[] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] & [] \\
[] & [] & [] & [] & [] & [] & [] & G_{u31} & G_{u32} & [] & G_{u33} & G_{u34} \\
[] & [] & [] & [] & [] & [] & [] & G_{u41} & G_{u42} & [] & G_{u43} & G_{u44}
\end{bmatrix}$$

$$\mathbf{G}_{se}(f) = \mathbf{k}_e \cdot \mathbf{G}_{ue}(f) \cdot \mathbf{k}_e^T$$

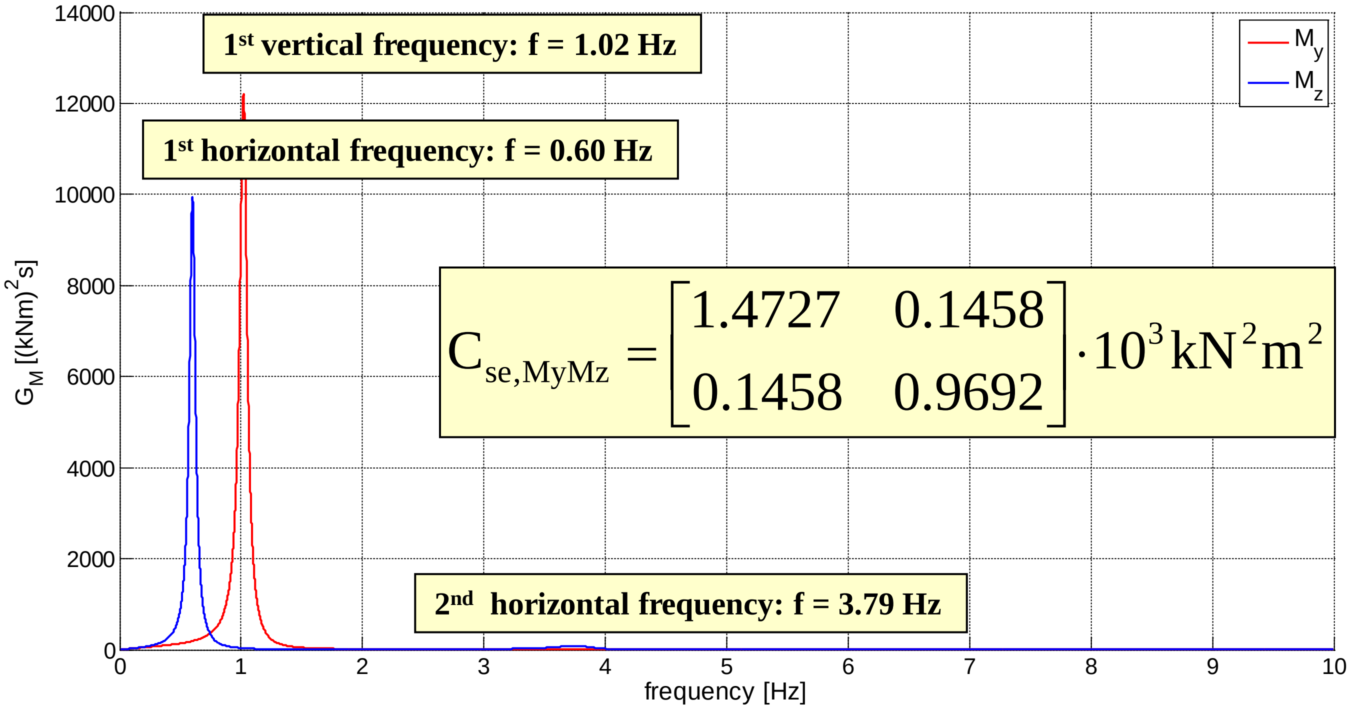
$$\mathbf{C}_{se} = \int_0^{f_{max}} \mathbf{G}_{se}(f)\,df$$

$$\sigma_{My}^2 = \mathbf{C}_{se}(5,5)$$

$$\sigma_{Mz}^2 = \mathbf{C}_{se}(6,6)$$

Auto-Spectra of the Bending Moments M_y and M_z

The auto-spectra of the bending moments are dominated by the 1st modes. The 2nd horizontal bending mode has a small, negligible influence on the moment M_y. The integral of the cross-spectral matrix shows a marked de-correlation between M_y und M_z: the maximum values do not occur simultaneously!



$$C_{se,MyMz} = \begin{bmatrix} 1.4727 & 0.1458 \\ 0.1458 & 0.9692 \end{bmatrix} \cdot 10^3 \, kN^2 m^2$$

Cross-Variances of the Stress Resultants

The spectral matrix of the element forces is computed by multiplying the spectral matrix of the element displacement with the element stiffness matrix. The stiffness matrix does not depend on the frequency.

$$\mathbf{G}_{se}(f) = \mathbf{k}_e \cdot \mathbf{G}_{ue}(f) \cdot \mathbf{k}_e^T$$

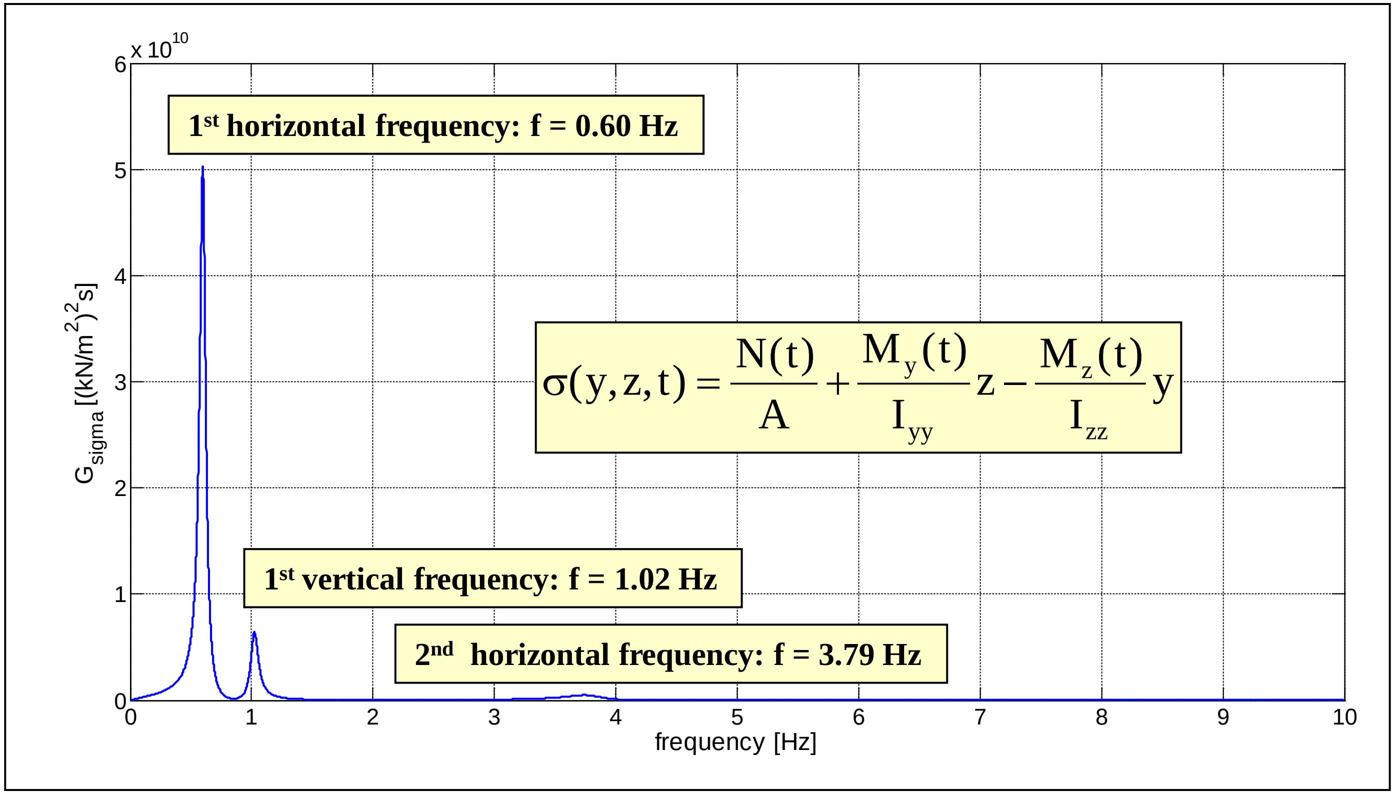
So when integrating the spectral matrix of the element forces to obtain the variances, we can extract the stiffness matrix from the integral:

$$\mathbf{C}_{se} = \int_0^{f_{max}} \mathbf{k}_e \cdot \mathbf{G}_{ue}(f) \cdot \mathbf{k}_e^T df = \mathbf{k}_e \cdot \int_0^{f_{max}} \mathbf{G}_{ue}(f) df \cdot \mathbf{k}_e^T = \mathbf{k}_e \cdot \mathbf{C}_{ue} \cdot \mathbf{k}_e^T$$

As a consequence it is not necessary to construct the frequency-dependent spectral matrix of the element forces (only if we want to plot the spectra). Instead we can directly use the matrix C_u of the cross-variances (this is just a “normal” matrix with scalar elements) wich we have already computed to obtain the standard deviations of the system degrees of freedom and perform the element computation directly for the already integrated matrices. In principle it would be possible to delete $G_u(f)$ once it has been integrated to C_u.

Auto-Spectrum of the Normal Stress σ_{xx}

The normal stress σ_{xx} depends on both bending moments. Therefore all relevant eigenmodes can be seen in the auto-spectrum. Again, if we are not interested in the plot, we could compute the variance of σ_{xx} directly from C_{se}.



$$\sigma(y,z,t) = \frac{N(t)}{A} + \frac{M_y(t)}{I_{yy}} z - \frac{M_z(t)}{I_{zz}} y$$

Comparison Time Domain – Spectral Domain

The *standard deviations* computed in the spectral domain can be regarded as the true values since the loading is only given as spectral loading. The time histories are derived from the true spectra by a numerical procedure which introduces errors on several levels (the generation procedure, the time integration, the limited length of the histories).

The comparison in the table below shows a good agreement. From this we can deduce that both approaches and their implementations are correct and yield, within the unavoidable margin of error, equivalent results.

	standard deviations σ				
	u_{y2} [m]	u_{z2} [m]	M_y [kNm]	M_z [kNm]	σ_{xx}[kN/m²]
TD	1.408e-02	6.078e-03	3.798e+01	3.087e+01	8.090e+04
FD	1.413e-02	6.134e-03	3.838e+01	3.113e+01	8.162e+04
error [%]	0.4	0.9	1.0	0.8	0.9

calculation of σ_{xx} at y = -b/2, z = h/2

Caveat!

Spectral results cannot be combined to computed further derived results! So it is not possible to compute the standard deviations of the bending moments M_y and M_z and from them the standard deviation of the normal stress σ_{xx}. This can be easily demonstrated by assuming a constant peak factor g and computing the resulting maximum normal stress.

maximum stress from maximum bending moments

$$M_{y,max} = g \cdot \sigma_{my} = g \cdot 38.38\,\text{kNm}$$

$$M_{z,max} = g \cdot \sigma_{mz} = g \cdot 31.13\,\text{kNm}$$

$$\sigma_{xx,max,max} = \frac{M_{y,max}}{I_{yy}}\frac{h}{2} + \frac{M_{z,max}}{I_{zz}}\frac{b}{2} = g \cdot 103057\,\frac{\text{kN}}{\text{m}^2}$$

$$\varepsilon_{error} = \frac{103507 - 81618}{81618}100 = 26\,\%$$

maximum stress from standard deviation and peak faktor

$$\sigma_{xx,max,true} = g \cdot \sigma_{\sigma xx} = g \cdot 81618\,\frac{\text{kN}}{\text{m}^2}$$

The stress calculated from $M_{y,max}$ and $M_{z,max}$ assumes full correlation of these two moments, i.e. $M_{y,max}$ and $M_{z,max}$ are supposed to occur at the same time instance. That, however, is not true: the cross-variances $C_{se,MyMz}$ showed a marked de-correlation. The incorrect non–consideration of the de-correlation overestimates the dynamic part of the maximum stress by 26 %!

Limitation of the Spectral Approach

The spectral approach requires linear behavior at each step of the calculation. Many equations in structural mechanics, however, are nonlinear:

- Aerodynamics:
  - The wind pressure depends quadratically on the wind speed.
  - Angles of attack are computed by the nonlinear sine/cosine functions.
  - Aerodynamic coefficients depends empirically from the angle of attack.
- Equivalent von Mises stresses are defined by a nonlinear formula
- ….

In all these cases the problem must be linearized which introduces modeling errors which add up during the resulting chain of subsequent linearizations.

Also for some problems forces are needed and not stresses. The design of an R/C beam. e.g., requires the normal force and the bending moment as separate values. The cross-sectional design (k_d-method in Germany) is a nonlinear black-box method which cannot be formulated in the spectral domain. So we only know the maximum values for N and M, and we know that they do not occur at the same time. Yet there exist no rules for defining design combinations (such as: design for M_{max} and αN_{max}, with α being some correlation-dependent combination factor). Here we come up against a severe limitation of the spectral approach.

Application III:
Bogibeel-Bridge India

General Information on the Project

Unofficial unauthorized information: currently unavailable!

Modal Analysis of MDOF Systems in the Spectral Domain

Under construction …..