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Lecture Series:
Structural Dynamics

Lecture 12:

Stochastic Excitation

Part B: Response to Stochastic Loading



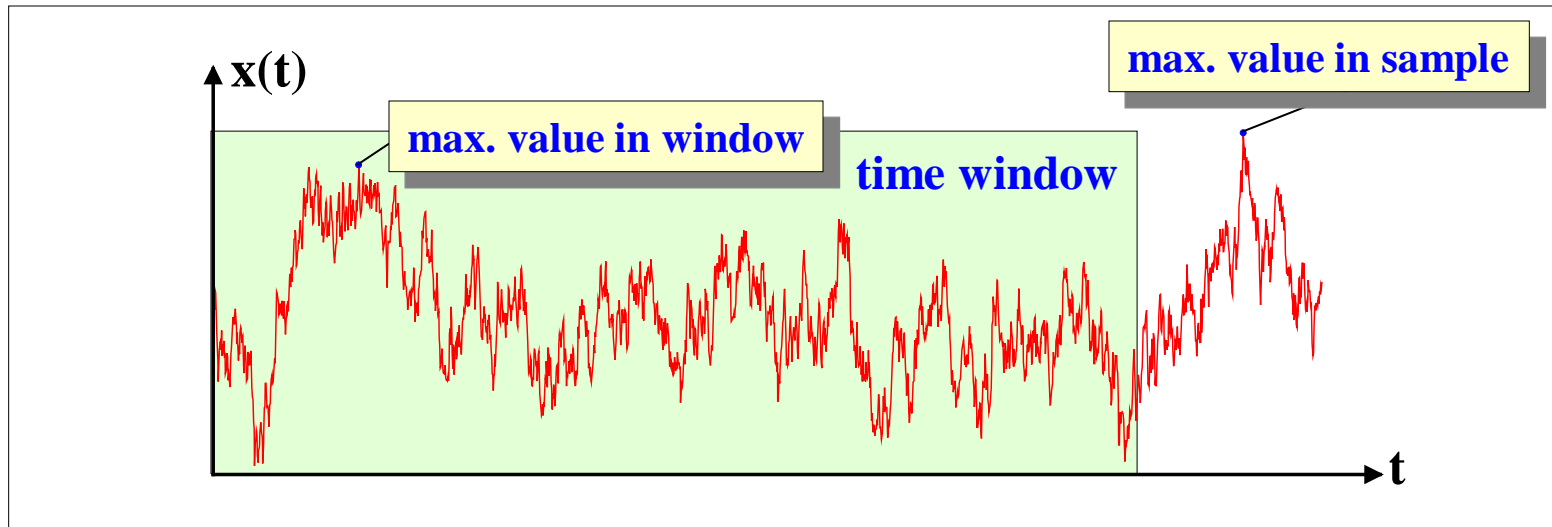
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Overview

- **Maximum expected values**
- **Solution in the spectral domain**
 - **auto-spectrum of the response**
 - **admittance functions**
 - **variance of the response**
- **Example: wind loading**
- **Outlook: a field of correlated load processes**



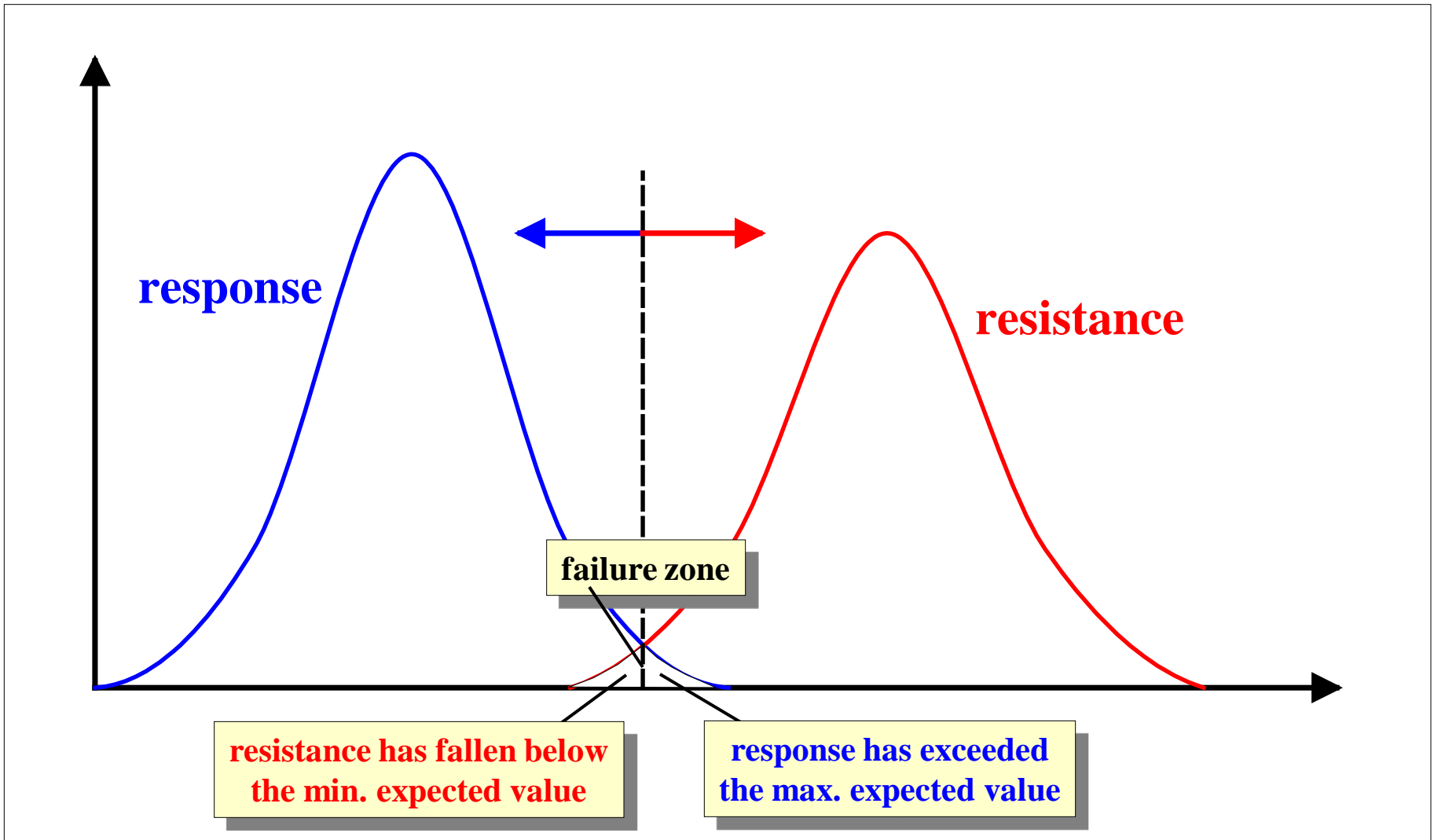
Maximum Values



- **Within a finite time window we can calculate the maximum value.**
- **There is no guarantee that there will not be a higher value outside the time window.**
- **There is no true maximum value; we can only determine a threshold that will only be surpassed by a given probability.**

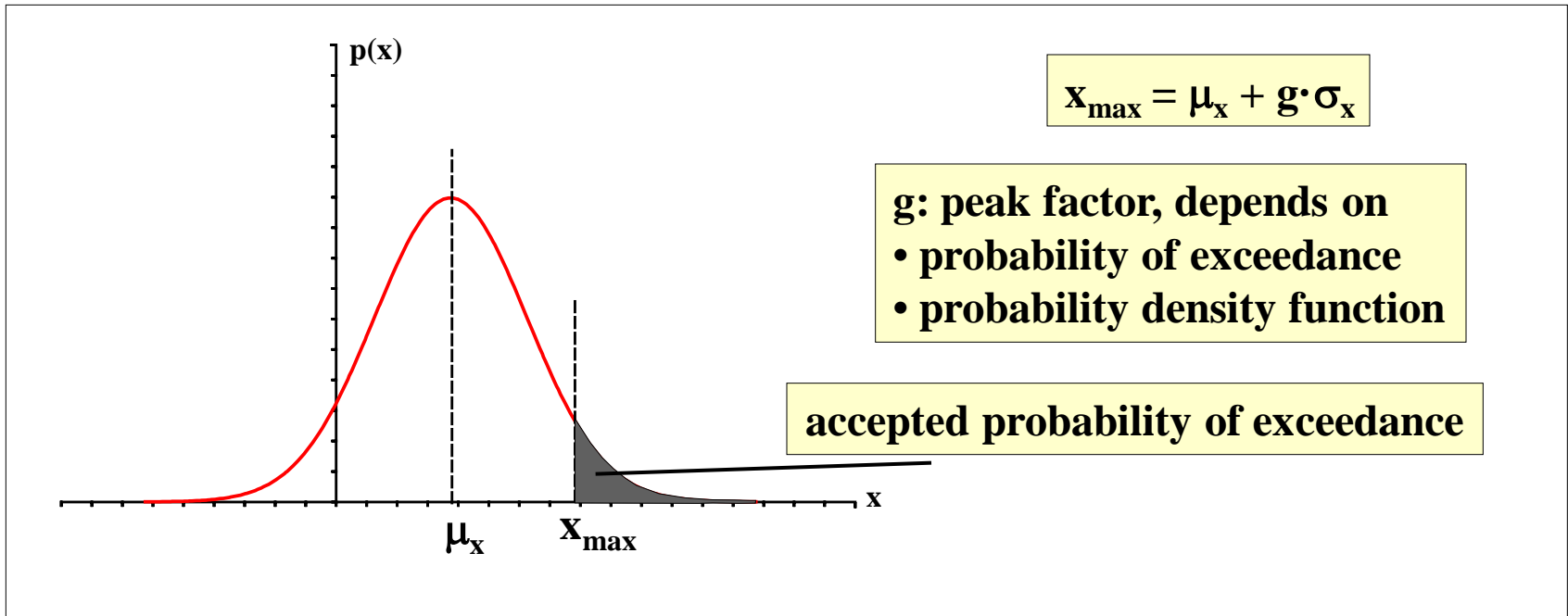


Probability of Failure



Maximum Expected Value

The maximum expected value is not a true maximum but a boundary value that can be surpassed. The probability of exceeding this value, however, is known and can be defined by the engineer. The boundary value will be used as the design value that the structure has to withstand.



Peak Factor for GAUSSIAN Processes

GAUSSIAN PDF:

$$p_{\text{gauss}}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

probability of exceedance:

$$P(x > x_{\text{max}}) = \int_{x_{\text{max}}}^{\infty} p_{\text{gauss}}(x) dx$$

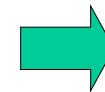
general GAUSSIAN distribution:

x has $N(\mu, \sigma)$ -distribution

normalized unit GAUSSIAN distribution:

$y = (x - \mu) / \sigma$ has $N(0, 1)$ -distribution

$$y = \frac{x - \mu_x}{\sigma_x}$$



$$x = \mu_x + y \cdot \sigma_x$$



Procedure for Determining the Peak Factor

Procedure

- (1) Choose P_{bound}
- (2) Determine y_{max} from the table so that $P(y \geq y_{\text{max}}) = P_{\text{bound}}$
- (3) Transform to general distribution:
 $x_{\text{max}} = \mu_x + \sigma_x y_{\text{max}} = \mu_x + \sigma_x g$
- (4) Peak factor: $g = y_{\text{max}}$

P_{bound}	g
5 %	1.645
1%	2.33
0.28 %	2.80
0.23 %	3.5



Example: Probability of Exceedance 3 %

Werte $\Phi(x)$ der Verteilungsfunktion der $N(0,1)$ -Verteilung

$x \setminus *$	0	1	2	3	4	5	6	7	8	9
0,0*	0,50000	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,52790	0,53188	0,53586
0,1*	0,50399	0,50798	0,51197	0,51595	0,51994	0,52392	0,52790	0,53188	0,53586	0,53984
0,2*	0,50798	0,51197	0,51595	0,51994	0,52392	0,52790	0,53188	0,53586	0,53984	0,54382
0,3*	0,51197	0,51595	0,51994	0,52392	0,52790	0,53188	0,53586	0,53984	0,54382	0,54780
0,4*	0,51595	0,51994	0,52392	0,52790	0,53188	0,53586	0,53984	0,54382	0,54780	0,55178
0,5*	0,51994	0,52392	0,52790	0,53188	0,53586	0,53984	0,54382	0,54780	0,55178	0,55576
0,6*	0,52392	0,52790	0,53188	0,53586	0,53984	0,54382	0,54780	0,55178	0,55576	0,55974
0,7*	0,52790	0,53188	0,53586	0,53984	0,54382	0,54780	0,55178	0,55576	0,55974	0,56372
0,8*	0,53188	0,53586	0,53984	0,54382	0,54780	0,55178	0,55576	0,55974	0,56372	0,56770
0,9*	0,53586	0,53984	0,54382	0,54780	0,55178	0,55576	0,55974	0,56372	0,56770	0,57168
1,0*	0,53984	0,54382	0,54780	0,55178	0,55576	0,55974	0,56372	0,56770	0,57168	0,57566
1,1*	0,54382	0,54780	0,55178	0,55576	0,55974	0,56372	0,56770	0,57168	0,57566	0,57964
1,2*	0,54780	0,55178	0,55576	0,55974	0,56372	0,56770	0,57168	0,57566	0,57964	0,58362
1,3*	0,55178	0,55576	0,55974	0,56372	0,56770	0,57168	0,57566	0,57964	0,58362	0,58760
1,4*	0,55576	0,55974	0,56372	0,56770	0,57168	0,57566	0,57964	0,58362	0,58760	0,59158
1,5*	0,55974	0,56372	0,56770	0,57168	0,57566	0,57964	0,58362	0,58760	0,59158	0,59556
1,6*	0,56372	0,56770	0,57168	0,57566	0,57964	0,58362	0,58760	0,59158	0,59556	0,59954
1,7*	0,56770	0,57168	0,57566	0,57964	0,58362	0,58760	0,59158	0,59556	0,59954	0,60352
1,8*	0,57168	0,57566	0,57964	0,58362	0,58760	0,59158	0,59556	0,59954	0,60352	0,60750
1,9*	0,57566	0,57964	0,58362	0,58760	0,59158	0,59556	0,59954	0,60352	0,60750	0,61148
2,0*	0,57964	0,58362	0,58760	0,59158	0,59556	0,59954	0,60352	0,60750	0,61148	0,61546
2,1*	0,58362	0,58760	0,59158	0,59556	0,59954	0,60352	0,60750	0,61148	0,61546	0,61944
2,2*	0,58760	0,59158	0,59556	0,59954	0,60352	0,60750	0,61148	0,61546	0,61944	0,62342
2,3*	0,59158	0,59556	0,59954	0,60352	0,60750	0,61148	0,61546	0,61944	0,62342	0,62740
2,4*	0,59556	0,59954	0,60352	0,60750	0,61148	0,61546	0,61944	0,62342	0,62740	0,63138
2,5*	0,59954	0,60352	0,60750	0,61148	0,61546	0,61944	0,62342	0,62740	0,63138	0,63536
2,6*	0,60352	0,60750	0,61148	0,61546	0,61944	0,62342	0,62740	0,63138	0,63536	0,63934
2,7*	0,60750	0,61148	0,61546	0,61944	0,62342	0,62740	0,63138	0,63536	0,63934	0,64332
2,8*	0,61148	0,61546	0,61944	0,62342	0,62740	0,63138	0,63536	0,63934	0,64332	0,64730
2,9*	0,61546	0,61944	0,62342	0,62740	0,63138	0,63536	0,63934	0,64332	0,64730	0,65128
3,0*	0,61944	0,62342	0,62740	0,63138	0,63536	0,63934	0,64332	0,64730	0,65128	0,65526
3,1*	0,62342	0,62740	0,63138	0,63536	0,63934	0,64332	0,64730	0,65128	0,65526	0,65924
3,2*	0,62740	0,63138	0,63536	0,63934	0,64332	0,64730	0,65128	0,65526	0,65924	0,66322
3,3*	0,63138	0,63536	0,63934	0,64332	0,64730	0,65128	0,65526	0,65924	0,66322	0,66720
3,4*	0,63536	0,63934	0,64332	0,64730	0,65128	0,65526	0,65924	0,66322	0,66720	0,67118
3,5*	0,63934	0,64332	0,64730	0,65128	0,65526	0,65924	0,66322	0,66720	0,67118	0,67516
3,6*	0,64332	0,64730	0,65128	0,65526	0,65924	0,66322	0,66720	0,67118	0,67516	0,67914
3,7*	0,64730	0,65128	0,65526	0,65924	0,66322	0,66720	0,67118	0,67516	0,67914	0,68312
3,8*	0,65128	0,65526	0,65924	0,66322	0,66720	0,67118	0,67516	0,67914	0,68312	0,68710
3,9*	0,65526	0,65924	0,66322	0,66720	0,67118	0,67516	0,67914	0,68312	0,68710	0,69108
4,0*	0,65924	0,66322	0,66720	0,67118	0,67516	0,67914	0,68312	0,68710	0,69108	0,69506

$$\Phi(x) = \int_{-\infty}^x p(x) dx$$

Probability of exceedance: 3 %



$$\Phi(x) = 1 - 0.03 = 0.97$$



from table

$$\Phi(x = 1.88) = 0.96995$$

$$\Phi(x = 1.89) = 0.97062$$



on the safe side

$$g = 1.89$$



Procedure for Design

- (1) Determine μ and σ of the design variable v .
- (2) Define probability P_{bound} .
- (3) Calculate peak factor g .
- (4) Calculate design value for v : $v_{\text{max}} = \mu_v + g \cdot \sigma_v$

Mean value:

The mean value of the load leads to the mean value of the response. The dynamic response fluctuates about this mean. So the mean response can be calculated by a static analysis.

RMS:

The fluctuations of the load lead to the dynamic excitations. The closer the peaks of the load spectrum S_{load} coincide with the peaks of the admittance function, the larger the dynamic effects are!

Open question: how do we find the standard deviation of the response?



Review: Response of an SDOF-System

DUHAMEL integral for arbitrary loading:

$$v(t) = \int_0^t p(\tau) h(t - \tau) d\tau$$

$h(t-\tau)$: impulse reaction function, describes the influence of an impulse which took place τ time units before time t .

undamped case:

$$h(t) = \frac{1}{m\omega} \sin \omega t$$

damped case:

$$h(t) = \frac{1}{m\omega_d} e^{-\xi\omega t} \sin \omega_d t$$

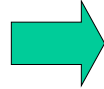


DUHAMEL Integral: Alternative Formulation

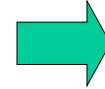
(a) Shift time lag from h to p :

Substitution:

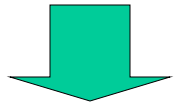
$$t - \tau = s$$



$$\tau = t - s$$



$$d\tau = -ds$$



$$v(t) = \int_{\tau=0}^{\tau=t} p(\tau) h(t - \tau) d\tau = - \int_{s=t}^{s=0} p(t - s) h(s) ds = \int_0^t p(t - s) h(s) ds$$

(b) Extend integration to negative time range:

$$v(t) = \int_{-\infty}^t p(t - s) h(s) ds$$

since $h(s) = 0$ for negative time values s



Step 1: Covariance of the Response

Definition:

$$R_v(\tau) = \lim_{T/2 \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t) v(t + \tau) dt$$

Substitute time response by DUHAMEL integral:

$$R_v(\tau) = \lim_{T/2 \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left\{ \int_{-\infty}^t p(t-r) h(r) dr \right\} \left\{ \int_{-\infty}^{t+\tau} p(t+\tau-s) h(s) ds \right\} dt$$

Incorporate the limes into the integration boundaries:

$$R_v(\tau) = \lim_{T/2 \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left\{ \int_{-\infty}^{\infty} p(t-r) h(r) dr \right\} \left\{ \int_{-\infty}^{\infty} p(t+\tau-s) h(s) ds \right\} dt$$



Re-order integrals with respect to t, r, s:

$$R_v(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(r)h(s) \left\{ \lim_{T/2 \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t-r)p(t+\tau-s) dt \right\} dr ds$$

Substitution in dt integral:

$$t - r = u \quad \rightarrow \quad t = u + r \quad \rightarrow \quad dt = du$$

$$t + \tau - s = u + r + \tau - s$$

Express dt integral as covariance function of load p:

$$\lim_{T/2 \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} p(u)p(u + \tau + r - s) du = R_p(\tau + r - s)$$



Result 1: Covariance of the Response

Final expression for the covariance of the response:

$$R_v(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(r) h(s) R_p(\tau + r - s) dr ds$$

The auto-covariance function of the structural response can be computed directly from the auto-covariance function of the load. The computation involves a double integral over the entire time domain from minus infinity to plus infinity. The structural properties (mass, damping, stiffness) enter the calculation via the impulse reaction function. That means that the knowledge of the response of a system to an impulse encompasses the full information regarding its dynamic properties.



Step 2: Auto-Spectrum of the Response

Definition of the auto-spectrum:

$$S_v(\Omega) = \int_{-\infty}^{\infty} R_v(\tau) e^{-i\Omega\tau} d\tau$$

Substitute covariance of the response by covariance of the load:

$$S_v(\Omega) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(r) h(s) R_p(\tau + r - s) dr ds \right\} e^{-i\Omega\tau} d\tau$$

Substitution:

$$\tau + r - s = w$$



$$\tau = w - r + s$$



$$d\tau = dw$$



Perform substitution:

$$S_v(\Omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(r) h(s) R_p(w) e^{-i\Omega(w-r+s)} dr ds dw$$

Re-order variables into 3 separate integrals:

$$S_v(\Omega) = \int_{-\infty}^{\infty} h(r) e^{i\Omega r} dr \cdot \int_{-\infty}^{\infty} h(s) e^{-i\Omega s} ds \cdot \int_{-\infty}^{\infty} R_p(w) e^{-i\Omega w} dw$$

Interpret integrals as FOURIER integrals:

$$S_v(\Omega) = \tilde{\underline{H}}(\Omega) \underline{H}(\Omega) S_p(\Omega) = |\underline{H}(\Omega)|^2 S_p(\Omega)$$



Mechanical Admittance

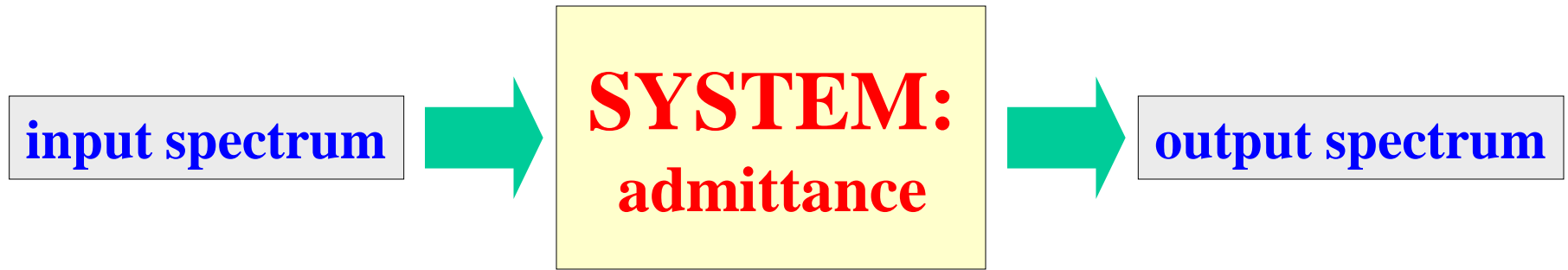
$$S_v(\Omega) = |\underline{H}(\Omega)|^2 S_p(\Omega) = \chi_{vp}(\Omega) S_p(\Omega)$$

The spectrum of the response can be directly obtained from the spectrum of the excitation by multiplication with the square of the absolute value of the FOURIER transform of the impulse reaction function.

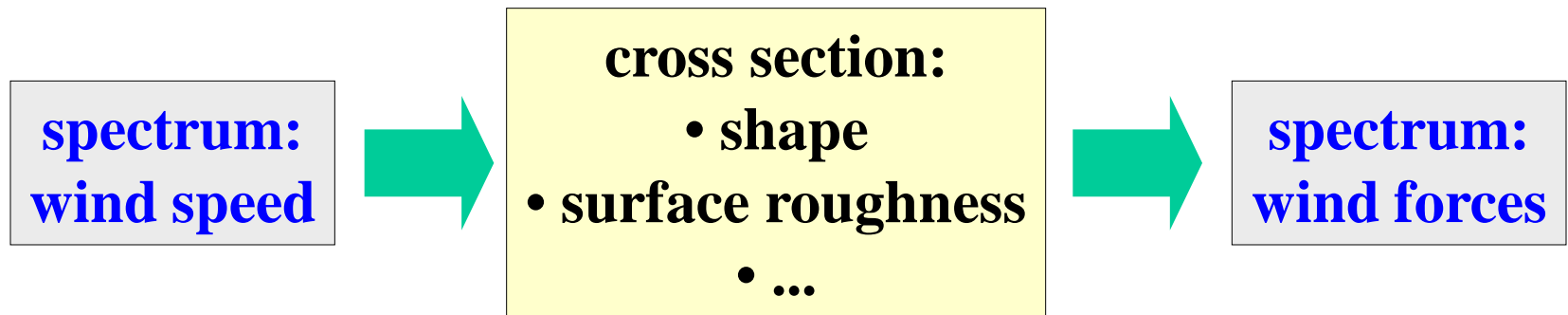
A function which transforms one auto-spectrum (input spectrum) into another (output spectrum) is called *admittance function*. In the present case we have the *mechanical admittance* which describes the admittance properties of the mechanical system. Since we know the impulse reaction function, we can easily derive the mechanical admittance by perform the FOURIER transformation analytically.



Admittance in General



Example: aerodynamic admittance:



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FOURIER Transformation of $h(t)$

Definition of the FOURIER transformation:

$$\underline{\underline{H}}(\Omega) = \int_{-\infty}^{\infty} h(t) e^{-i\Omega t} dt$$

Introduce $h(t)$ and note that $h(t) = 0$ for $t < 0$:

$$\underline{\underline{H}}(\Omega) = \int_0^{\infty} \frac{1}{m\omega_D} e^{-\xi\omega t} \sin \omega_D t e^{-i\Omega t} dt = \frac{1}{m\omega_D} \int_0^{\infty} e^{-(\xi\omega+i\Omega)t} \sin \omega_D t dt$$



From a mathematical handbook:

$$\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{e^{-ax}}{a^2 + b^2} (-a \sin bx - b \cos bx) \Big|_0^{\infty}$$



$$\int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2}$$

$$a = \xi\omega + i\Omega \quad b = \omega_D$$

Solution for the integral:

$$\underline{H}(\Omega) = \frac{1}{m\omega_D} \frac{\omega_D}{\omega_D^2 + (\xi\omega + i\Omega)^2}$$



$$\underline{H}(\Omega) = \frac{1}{m\{\omega^2 - \Omega^2 + i2\xi\omega\Omega\}}$$



(a) $\omega_D = \omega\sqrt{1 - \xi^2}$



(b) $\frac{\Omega}{\omega} = \eta$



FOURIER Transform H and Mechanical Admittance

$$\underline{H}(\Omega) = \frac{(1 - \eta^2) - i2\eta\xi}{k\{(1 - \eta^2)^2 + (2\eta\xi)^2\}}$$

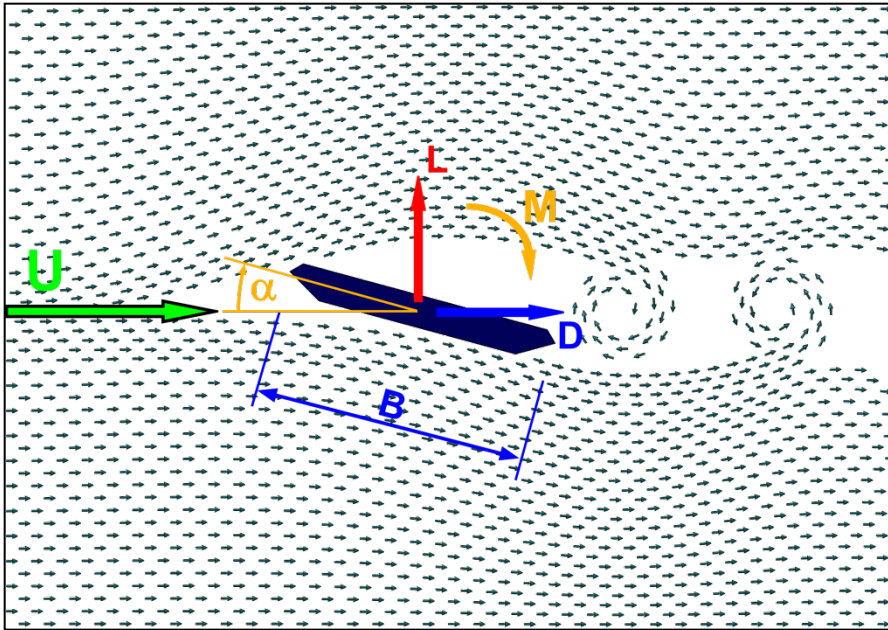
The FOURIER transform H of the impulse reaction function h is identical to the complex transfer function of the oscillator under harmonic excitation!

$$|\underline{H}|^2 = \text{Re}^2(\underline{H}) + \text{Im}^2(\underline{H}) = \frac{1}{k^2\{(1 - \eta^2)^2 + (2\eta\xi)^2\}}$$

The mechanical admittance is identical to the square of the dynamic amplification V_1 of the oscillator under harmonic excitation, divided by k^2 ! The information of the response to an impulse, known for all time instances, is equivalent to the knowledge of the maximum response under harmonic excitation for all frequencies!



Example: Quasi-Stationary Model for Wind Forces



We look at the wind-induced forces on a *unit strip* of the bridge deck. The wind flow around the cross section creates a pressure distribution on the boundary. These pressures can be integrated to obtain forces: the *drag force D*, the *lift force L* and the *moment M*. All forces are distributed forces per length unit of the bridge deck.

Their values depend on the wind speed and the shape of the cross section. The geometry of the cross section is captured by the three aerodynamic coefficients C_D , C_L and C_M , which can be measured in a boundary layer wind tunnel. The size of the cross section is expressed by the length B .

Force model:

$$D = q C_D(\alpha) B$$

$$L = q C_L(\alpha) B$$

$$M = q C_M(\alpha) B^2$$

$$q = \frac{1}{2} \rho U^2$$

variables:

- q : kinetic pressure
- ρ : density of air
- U : wind speed
- C_* : aerodynamic coefficients
- B : invested length



menu

Split into Static and Dynamic Effects

Total wind speed: mean value + fluctuating part:

$$U = \bar{U} + U'$$



$$F = \frac{1}{2} \rho C_* B (\bar{U}^2 + 2\bar{U}U' + U'^2) = \bar{F} + F'$$

mean load: static deformation

$$\bar{F} = \frac{1}{2} \rho C_* B \bar{U}^2$$

fluctuating load: dynamic deformation

$$F' = \rho C_* B \bar{U} U' + \frac{1}{2} \rho C_* B U'^2 \approx \rho C_* B \bar{U} U'$$

The fluctuating part has been linearized to allow a FD approach!



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Aerodynamic Admittance

Some thoughts on linear transformations:

$$\underline{y} = \lambda \underline{x} \quad \Rightarrow \quad \underline{Y}(\Omega) = \lambda \underline{X}(\Omega)$$

$$S_x(\Omega) = \lim_{T/2 \rightarrow \infty} \frac{1}{T} |\underline{X}(\Omega)|^2 \quad \Rightarrow \quad S_y(\Omega) = \lambda^2 S_x(\Omega)$$

Application to the present case:

$$S_F = \chi_{Fw} S_w$$

$$\chi_{Fw} = (\rho C_* B \bar{U})^2$$



Wind Spectrum of VON KÁRMÁN

Classic wind spectrum from literature:

$$S_w^* = \frac{\sigma_w^2}{f} \frac{4\bar{f}}{(1.0 + (8.409\bar{f})^2)^{5/6}}$$

$$\bar{f} = f \frac{L_w}{U_{10}}$$

parameters:

U_{10} : mean wind speed at 10 m height

L_w : integral length scale

σ_w : turbulence of the wind, $\sigma_w = I_w \cdot U_{10}$

turbulence intensity I_w :

- open ground: $I_w = 18$ %
- suburb, wood: $I_w = 31$ %
- city: $I_w = 56$ %

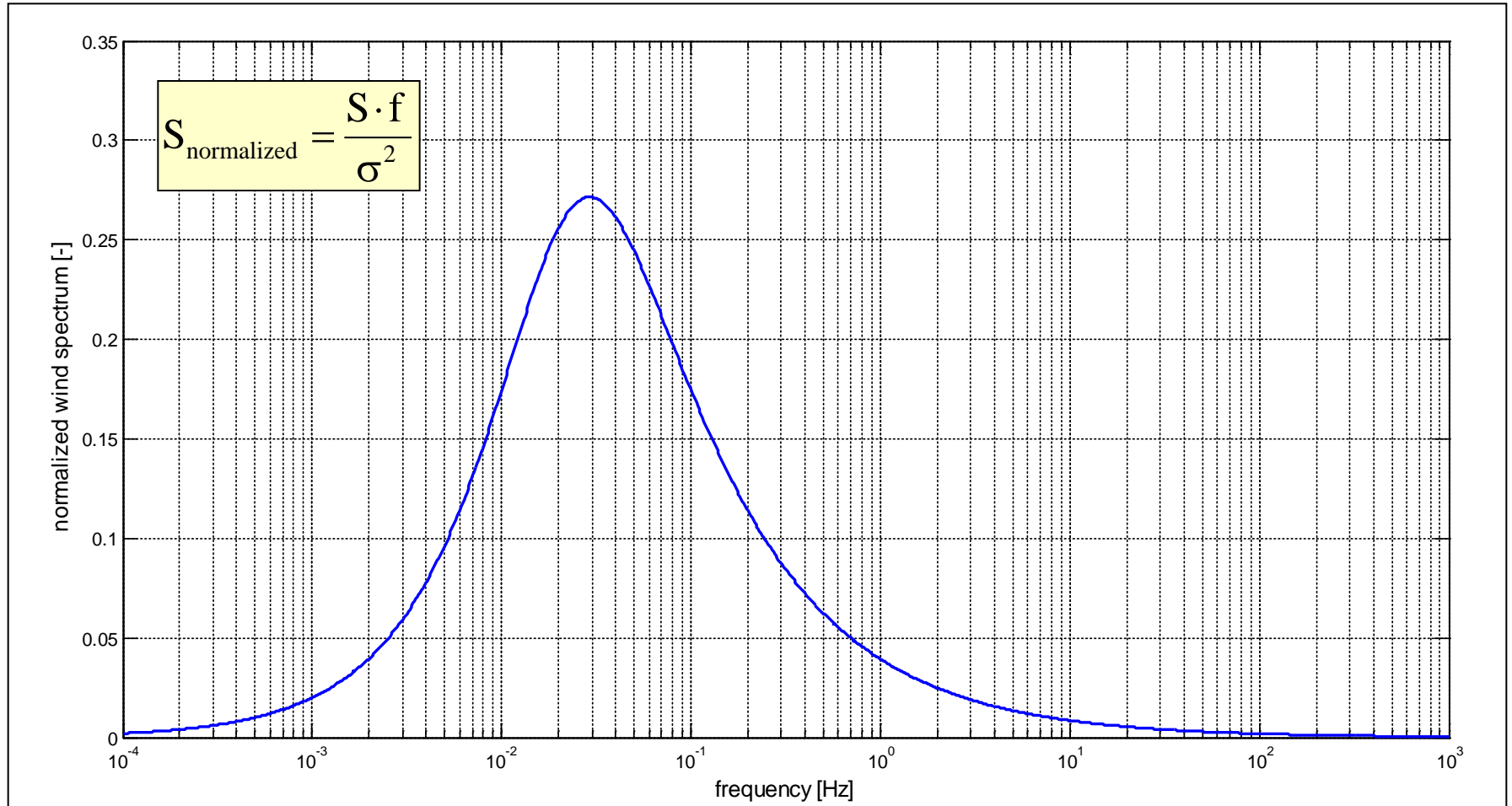
note!

$$\sigma_w^2 = \int_0^{\infty} S_w^*(f) df$$



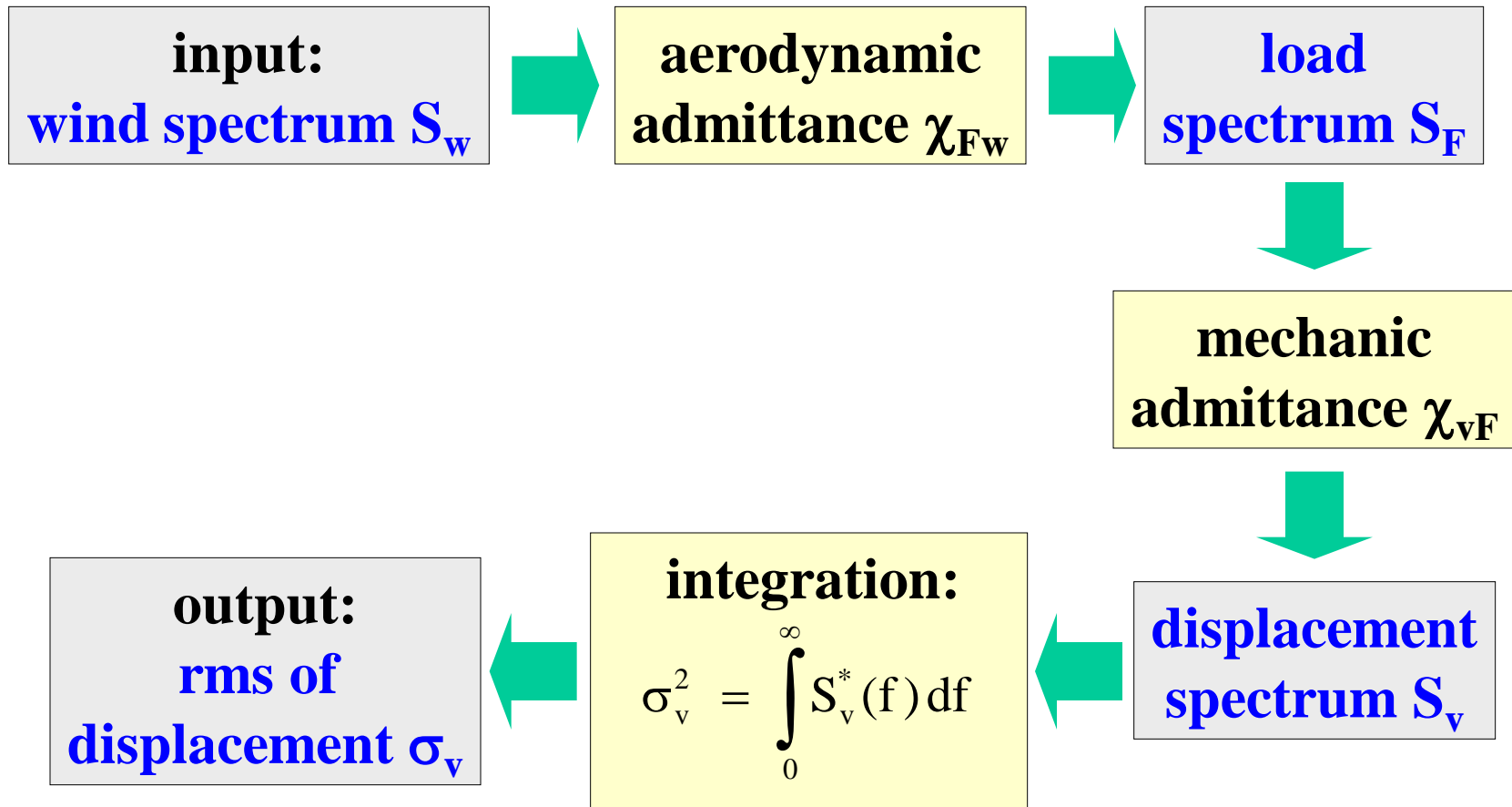
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Wind Spectrum of VON KÁRMÁN



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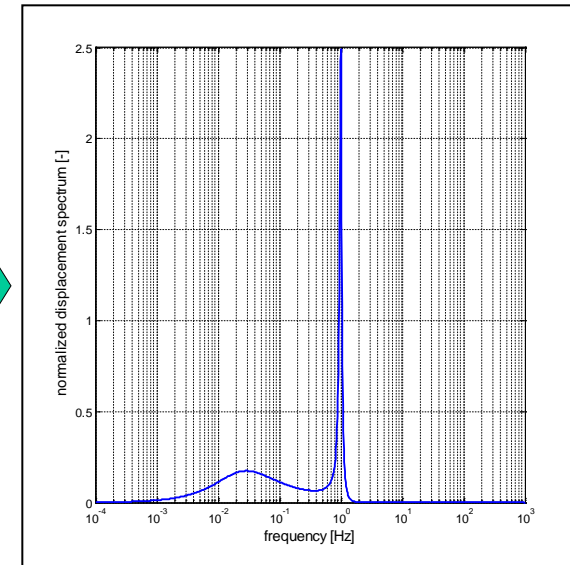
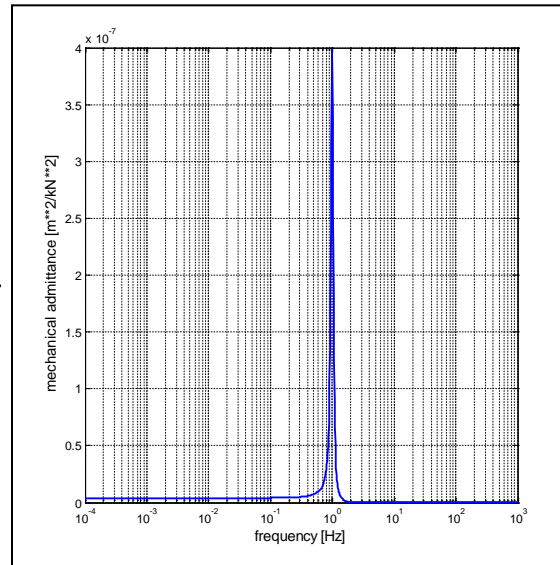
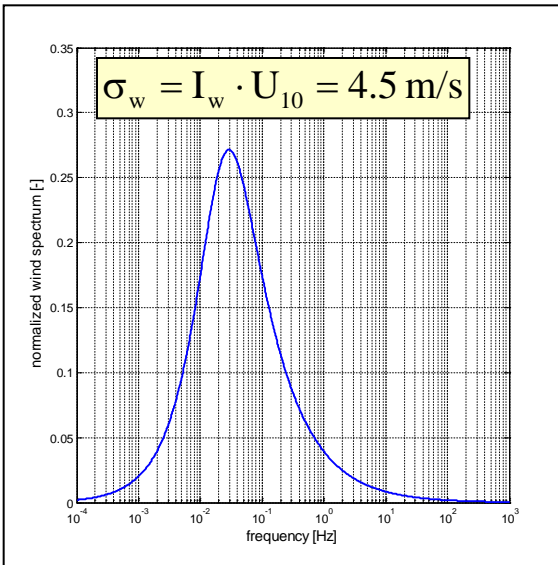
Flow Chart of the Calculation



Spectra for the Wind Problem

$L_w = 75.0 \text{ m}$
 $U_{10} = 15.0 \text{ m/s}$
 $I_w = 0.30$
 $U = 22.5 \text{ m/s}$

$k = 15791.0 \text{ (kN/m)/m}$
 $m = 400.0 \text{ (to/m)}$
 $\xi = 5 \%$
 $L = 10.0 \text{ m}$



$$\sigma_w = \sqrt{\int_0^{\infty} S_w(f) df} = 4.4954 \text{ m/s}$$



$$\sigma_v = \sqrt{\int_0^{\infty} S_v(f) df} = 0.14 \text{ mm}$$



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Spectral Approach and Time Domain Approach

The auto-spectrum characterizes the stochastic process per se. The solution in the spectral domain therefore produces the response per se: *one single calculation suffices* to capture the entire response in the spectral domain in the form of the auto-spectrum of the response. From the auto-spectrum we calculate the standard deviation of the response and our analysis is finished.

In the time domain we can only manifest the random process in realizations. There are special algorithms which generate realizations for a given process, i.e. for a given auto-spectrum.

For such a realization we can calculate a realization of the response by a time history analysis, e.g. with a direct time integration. Each realization, however, contains only partial information on the underlying random process. Therefore it is not possible to base a design on the results of only one single realization: a second realization might give larger results. Instead we have to generate a whole *batch of load realizations* and calculate a corresponding *batch of response realizations*. We then perform a suitable statistic analysis of our batch of results and define design values according to some chosen probability of failure.

For a computation of the wind response we typically perform 30 simulations. We find the design value by performing an *extreme value analysis* for the 30 individual extreme values. This is done by fitting an extreme value distribution, e.g. a *GUMBEL distribution*, into our computed 30 extrema and calculating from the fitted distribution the maximum expected value for a given probability of exceedance.

