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Lecture Series:

Structural Dynamics

Lecture 12:

Stochastic Excitation

Part A: Description of Stochastic Processes

Overview

- General remarks
- Properties of a single stochastic process:
  - statistic parameters
  - stochastic properties in the TD
  - stochastic properties in the FD
- Extension to fields of processes

Deterministic and Stochastic Processes

Process: evolution of a time-dependent variable x(t)

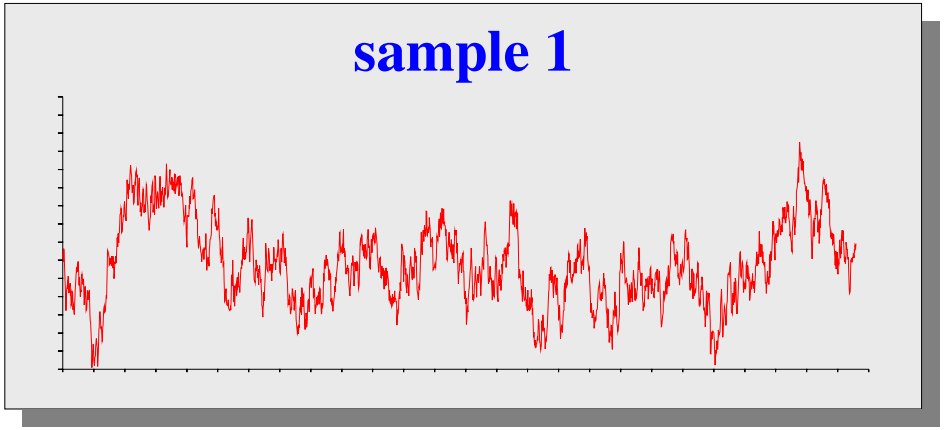
Deterministic process:
The value at time t can be predicted from a given relationship or from the values of the previous time instances.

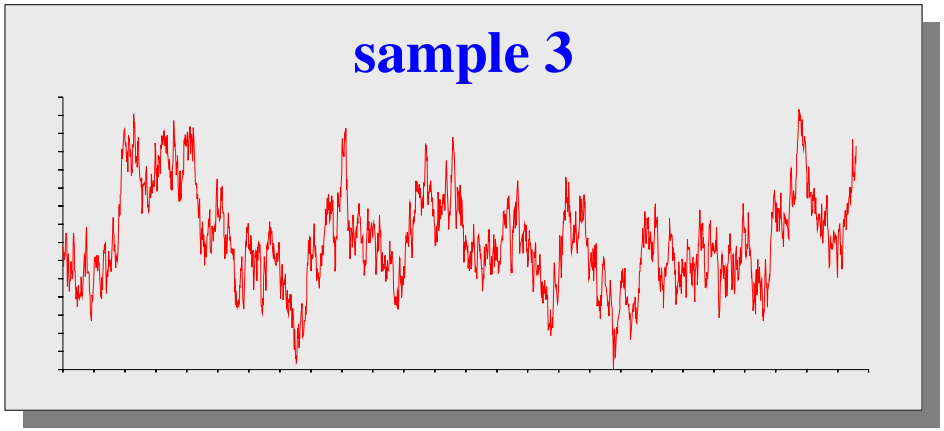
Stochastic process:
The value at time t cannot be predicted with certainty. Only some properties of the process are known.

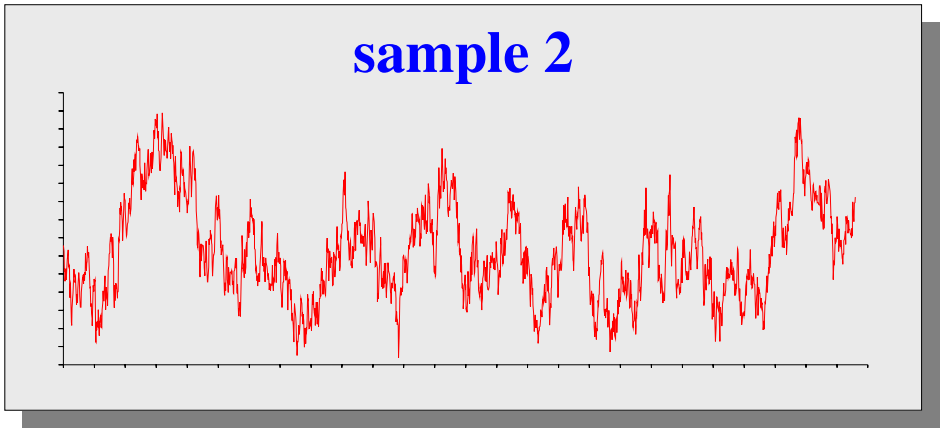
Process and Realisation

Process:

The *ensemble* of all possible measurements.
One single measurement: *realisation* or *sample* of the process.







Knowledge about the process is gained by individual measurements. Each measurement differs from all others – the measurements are *non-repeatable*. The process (the ensemble of all measurements) itself can only be *known partially* by its realisations.

Classification of Stochastic Processes

• stationary/instationary:

In a *stationary* process the characterising properties are time-independent within the time window under consideration.

• GAUSSIAN/Non-GAUSSIAN:

A *GAUSSIAN* process has a *GAUSSIAN probability density function*.

• Ergodic/Non-Ergodic:

One *single realisation* represents, apart from numerical inaccuracies, the *entire process*.

In the following:
Stationary, GAUSSIAN, ergodic processes.

Probability

In a *discrete process*, e.g. the casting of a die, we can assign a probability to each possible outcome. The probability of casting a 2 is exactly 1/6 for a perfect die. The assignment of a probability to a single-valued event is not possible for a *continuous real process* x(t).

Take e.g. the weight of chicken eggs. What is the probability of an egg having a weight of 80 grams? To find out we measure 10000 eggs with a letter scales with an accuracy of 1 gram and find that 100 eggs weigh 80 grams, so we would assign the 80-gram event a probability of 100/10000=0.01. Now someone presents us with a more accurate digital scales. We re-measure – better scales, better results – with the new scales which has a finer accuracy of 0.1 grams and find that only 10 of the original 100 eggs "really" weigh in at 80 grams – the rest have 79.8 or 79.9 or 80.2 or … grams. It was the limited accuracy of our first scales which erroneously assigned further 90 eggs the 80-gram weight. So our probability drops to 10/10000=0.001.

If we were to repeat the weight testing with an even more accurate scales, we would find even less eggs with a weight of 80.0 grams. In the limit of an infinitely accurate scales we would not find even a single egg with exactly the target weight. Between two real numbers it is always possible to fit in an infinity of further real numbers. So makes no sense to assign a probability to the exact outcome of a real random process – that probability is zero. Instead we define the *probability density function*.

Probability Density Function

Each possible value x of a random process x(t) is assigned a value p(x). The function p(x) is called *probability density function PDF*. The PDF does not describe the probability that the random process takes exactly the value x – it is only the *density of the probability*. The *probability proper* for the process x(t) is only defined for the event that the value x lies between two bounds x_a and x_b.

The probability P that x lies between 2 values x_a and x_b is given by:

$$P(x_a \leq x \leq x_b) = \int_{x_a}^{x_b} p(x)\,dx$$

The shape of the PDF depends on the physical nature of the specific process. It depends on certain *statistic parameters*.

Statistic Parameters

mean value:

$$\mu = E(x) = \int_{-\infty}^{+\infty} x\, p(x)\, dx$$

variance:

$$\sigma^2 = E[(x-\mu)^2] = \int_{-\infty}^{+\infty} x^2\, p(x)\, dx - \mu^2$$

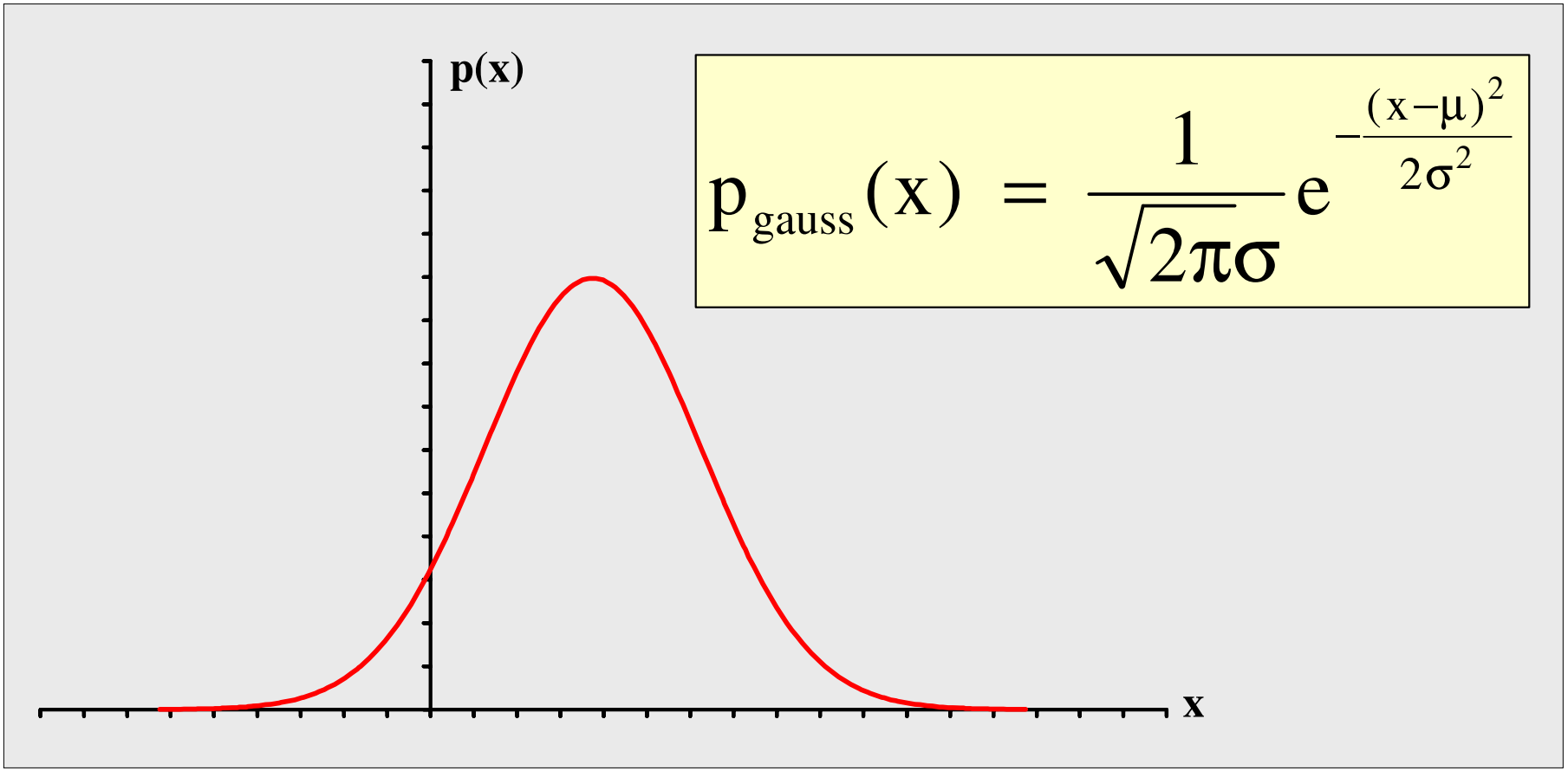
standard deviation or root mean square rms:

$$\sigma = \sqrt{\sigma^2}$$

There are further parameters depending on higher statistic moments (integrals of $x^n p(x)$) which do not interest here.

GAUSSIAN Probability Distribution

Processes which depend on a high number of independent variables can be described by a *GAUSSIAN PDF*, the so-called *GAUSSIAN bell curve*. It depends only on the mean value μ and the standard deviation σ.



$$p_{gauss}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Ergodicity

• Time mean:

Mean value of a single realisation (we average over time) $\Rightarrow \mu_t, \sigma_t$

• Ensemble mean:

Mean value of all realisations for a fixed time (we average over the ensemble) $\Rightarrow \mu_e, \sigma_e$

Condition for ergodicity:

$$\mu_t = \mu_e \, , \, \sigma_t = \sigma_e$$

Ergodic process: **The statistic parameters of the ensemble for a given time instance (for GAUSSIAN processes the mean value and the standard deviation) are identical to the ones computed by averaging over time for a given sample. Then one sample represents the entire process.**

Calculation of Statistic Parameters from a Finite Sample

We assume ergodicity: time mean is equivalent to ensemble mean.

continuous sample:

$$\mu_x = \frac{1}{T}\int_0^T x(t)\,dt$$

$$\sigma_x^2 = \frac{1}{T}\int_0^T x^2(t)\,dt - \mu_x^2$$

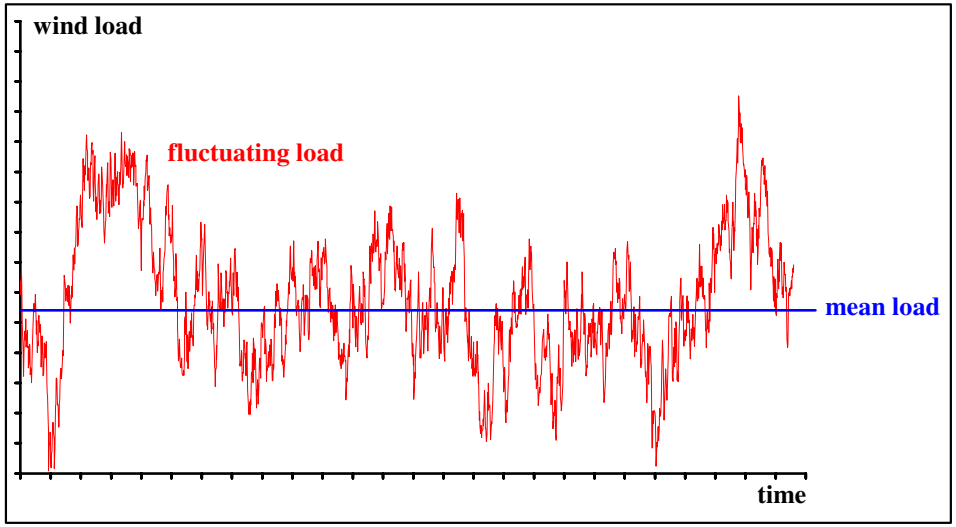
discrete sample:

$$\mu_x = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\sigma_x^2 = \frac{1}{N-1}\sum_{i=1}^{N} (x_i)^2 - \frac{N}{N-1}\mu_x^2$$

Dynamic Contents of a Signal

Natural load processes are characterized by a randomly fluctuating excitation whose fluctuations are often defined with respect to a mean value. The random excitation leads to a random vibration of the structure which can also be split into a mean part and a fluctuating part.

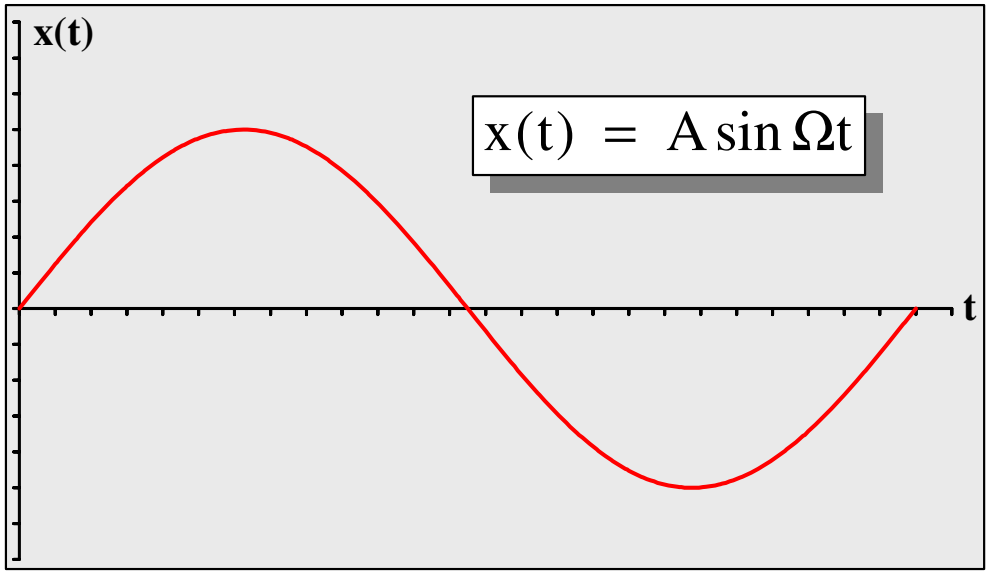


What would happen if the fluctuating part of the load were zero? Then we would have a constant load which would lead to a constant, i.e. static response, except for a short-lived transient phase caused by the load application. We can neglect this phase since we are interested in the long-time response. The dynamic response is therefore directly related to the fluctuations of the excitation.

The larger the fluctuations, the larger the dynamic response. How do we measure the *magnitude* of a fluctuation? The integral measure which captures not just the largest values which might occur only a few times but the *overall* magnitude is the *standard deviation σ*. The larger σ, the larger the dynamic contents of the signal. A σ of zero indicates a purely static excitation.

Standard Deviation of a Harmonic Signal

Harmonic functions are always of special interest in structural dynamics. So we start by analyzing the dynamic contents of a harmonic signal.

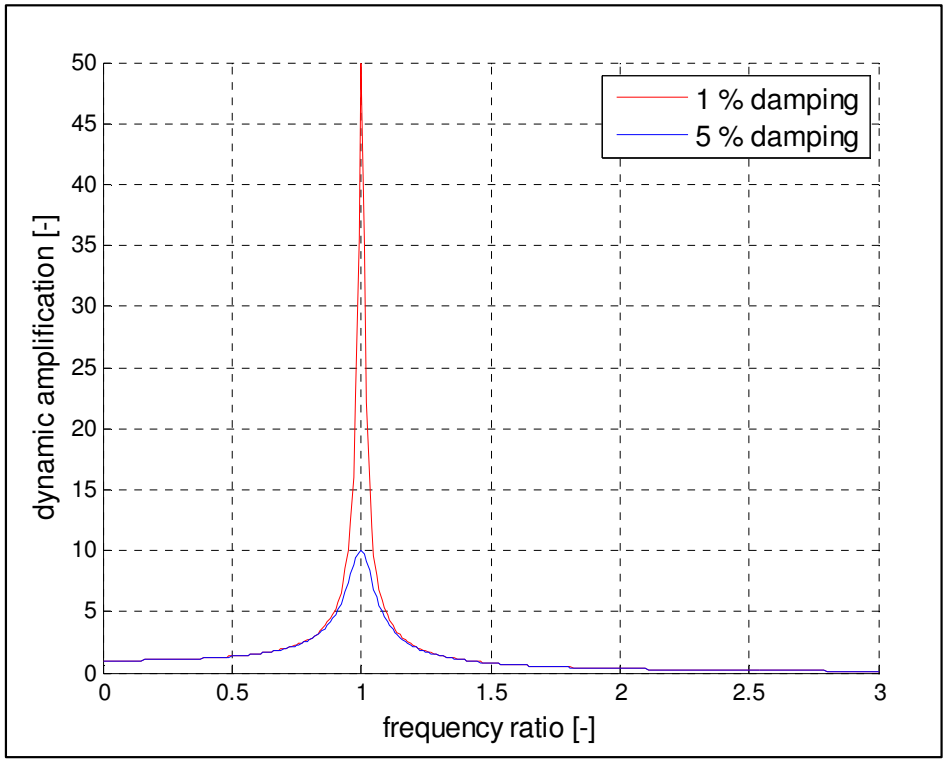


$$x(t) = A \sin \Omega t$$

$$\sigma_x^2 = \frac{1}{T} \int_{-T/2}^{T/2} (A \sin \Omega t)^2 \, dt = \frac{1}{2} A^2$$

We see that the standard deviation depends on the amplitude of the signal, but not on the frequency. All sinusoidal excitations have the same standard deviations, irrespective of their frequencies. But what about the structural response caused by the excitation? It also has a dynamic contents, and we ask ourselves if it is also independent from the load frequency. The answer is clearly no, as we know from the Dynamics I, lectures on harmonic and periodic loading.

Frequency-Dependence of the Response

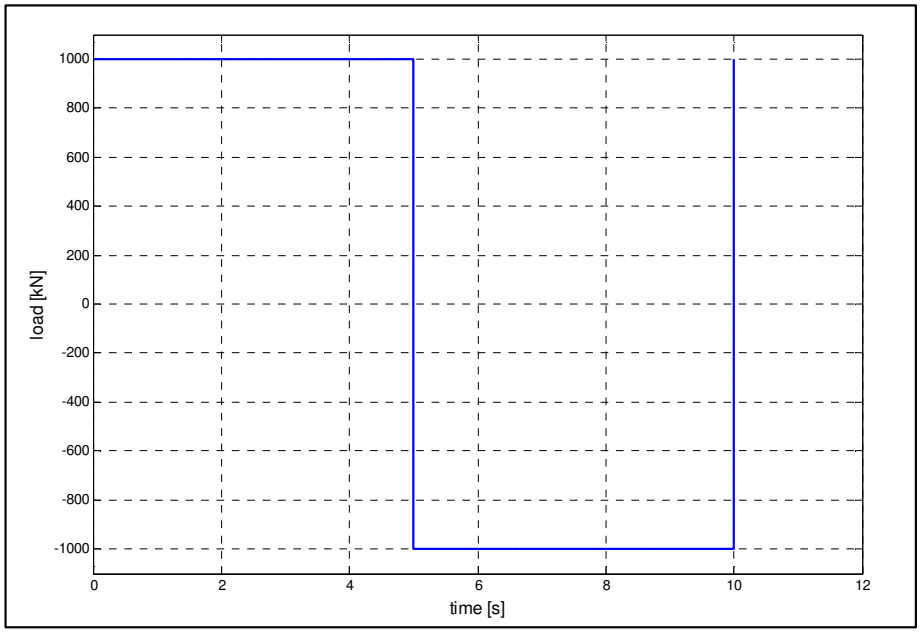


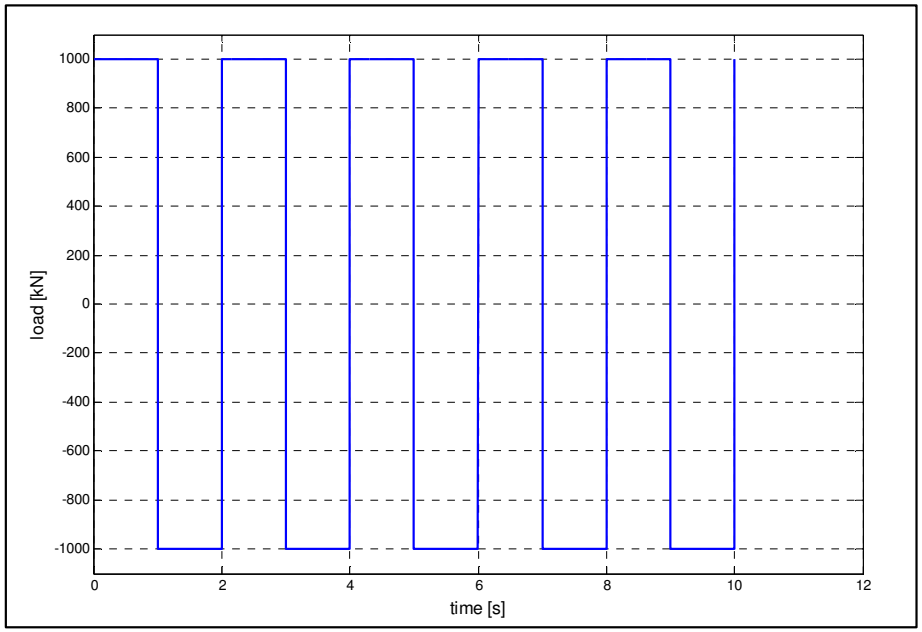
Statistic parameters alone do not characterize a stochastic process!

Time-dependent loading leads to dynamic amplification. This effect is captured for the case of a harmonic excitation with the amplification factor V_1, as shown above. The magnitude of the dynamic amplification is strongly influenced, one might say dominated, by the load frequency. In the case of resonance we observe the phenomenon that small load amplitudes can be amplified to such an extent that the structure cannot withstand the excitation.

Dynamic Response

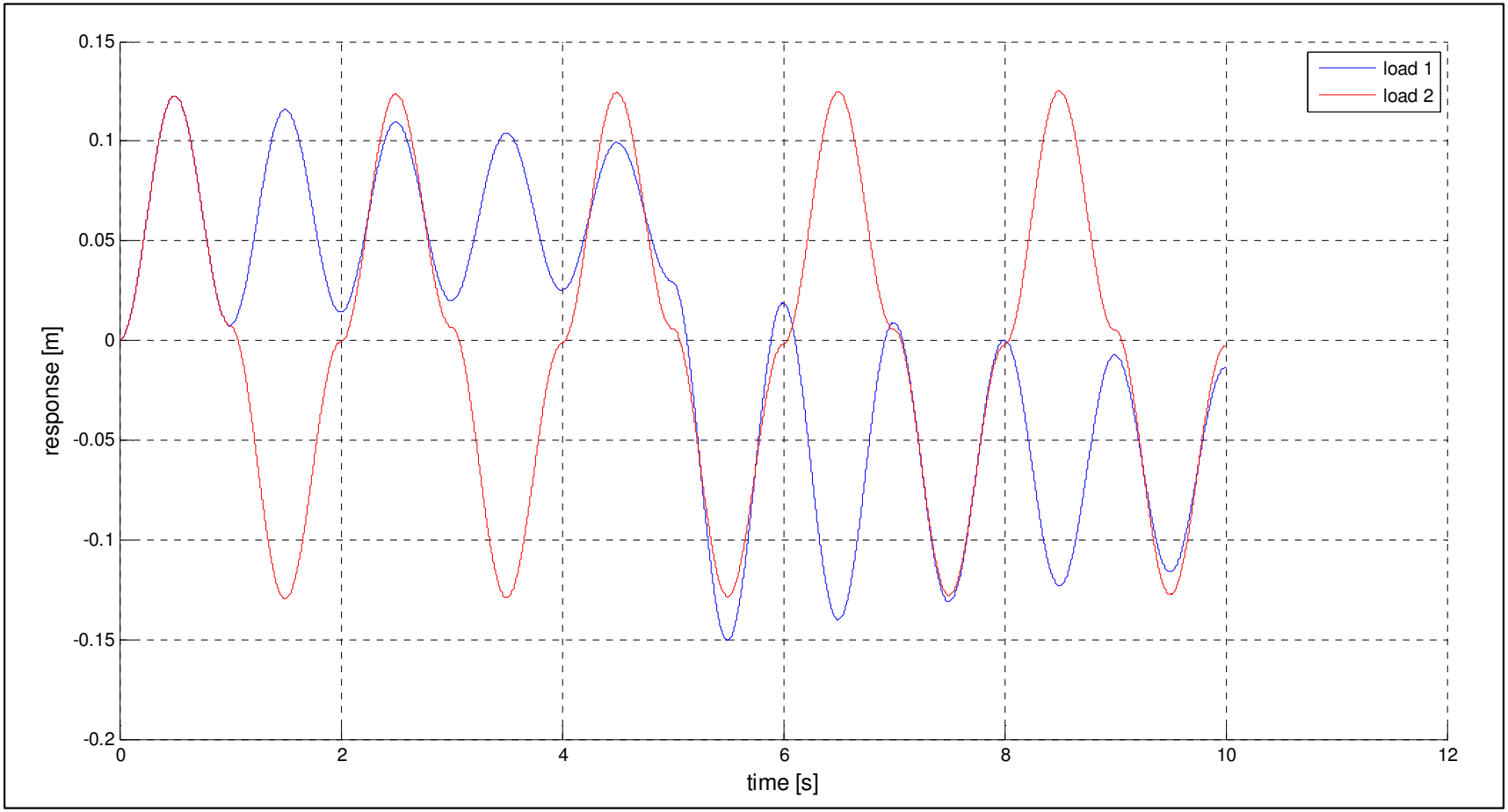
The statistic parameters depend only on the values of the signal, not on their sequence. The two load histories below fluctuate between -1000 and +1000. Both have an equal number of points with negative or positive values. Their standard deviations are therefore identical.





The sequence of the values, however, is not identical. History 1 has two phases with a constant load each, while history 2 is characterized by a saw-tooth that switches every 1.0 s between +1000 and -1000. The resulting response histories (an SDOF system with T = 1.0 s) show completely different response characteristics despite the identical rms of the excitation.

Dynamic Response



The *statistic parameters* allow a *static view* of the dynamic process. By averaging over time we eliminate the information regarding the sequence of events. To capture the dynamics of the process we need to include the sequence of events which leads us to *stochastic parameters*. These can be defined in both time and frequency domains.

Stochastic Properties in the TD

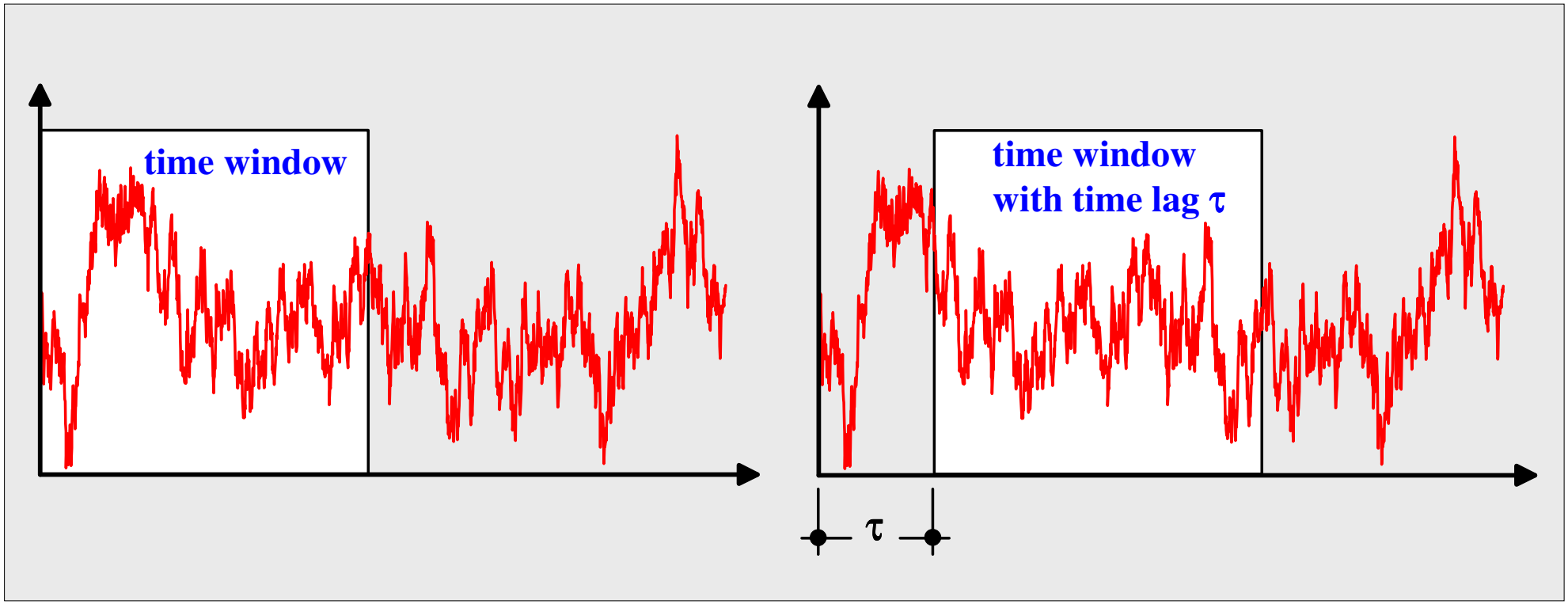
Definition:
Auto-covariance function **R(τ) for a process with *zero mean*:**

$$R(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x(t+\tau)\,dt \quad \Rightarrow \quad R(\tau=0) = \sigma^2$$

Auto-correlation function:

$$\rho(\tau) = \frac{R(\tau)}{\sigma^2} \quad \Rightarrow \quad \rho(\tau=0) = 1$$

Auto-Covariance Function



The auto-covariance function describes the internal structure of a process in the time domain. The "more deterministic" a process is, the stronger is the covariance within the process. We study several examples.

Example 1: Deterministic Process

harmonic process:

$$x(t) = A \sin \Omega t$$

The periodicity of x(t) is translated into R(τ)!

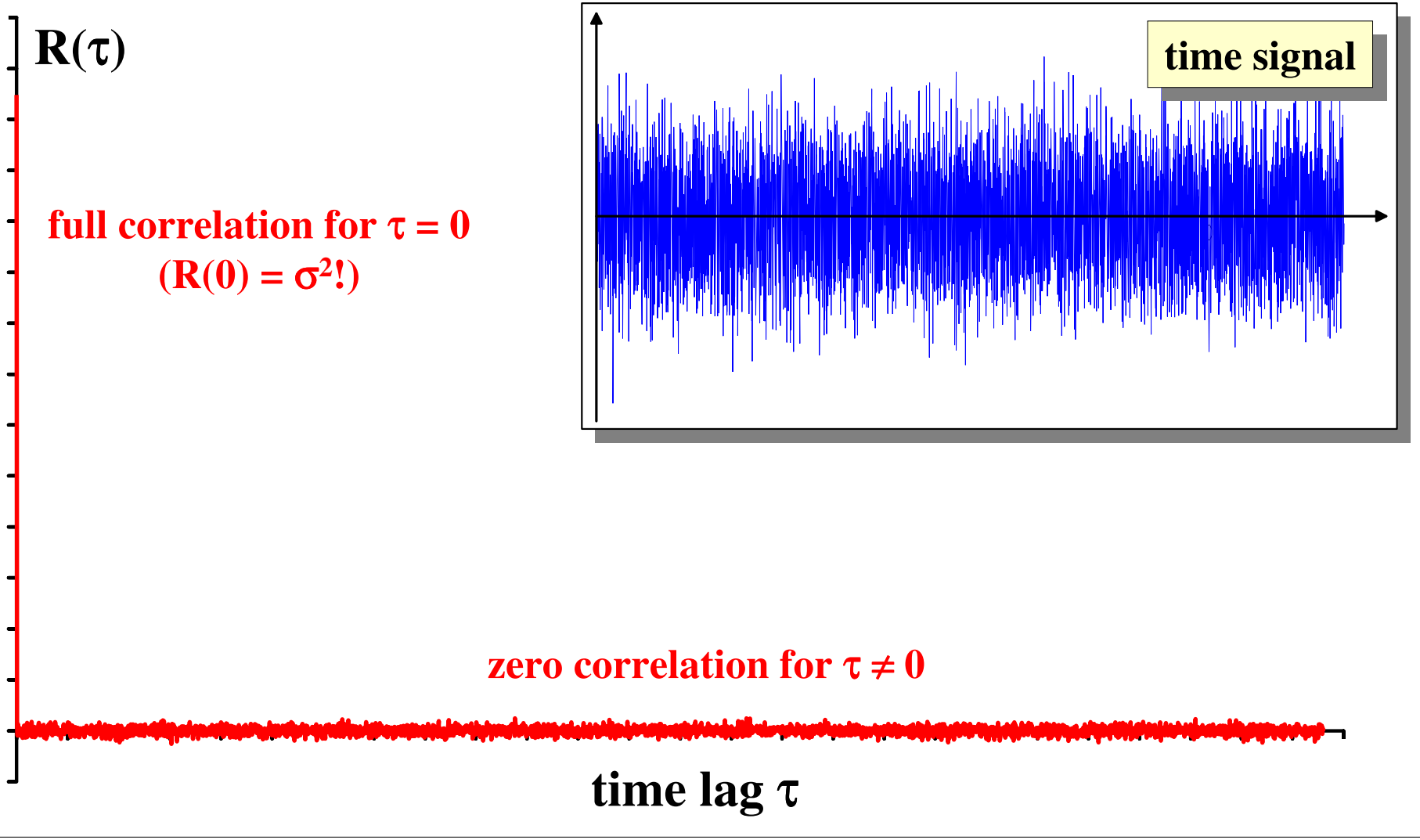
$$R(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} (A \sin \Omega t)(A \sin[\Omega(t+\tau)])\, dt$$

$$T = \frac{2\pi}{\Omega}$$

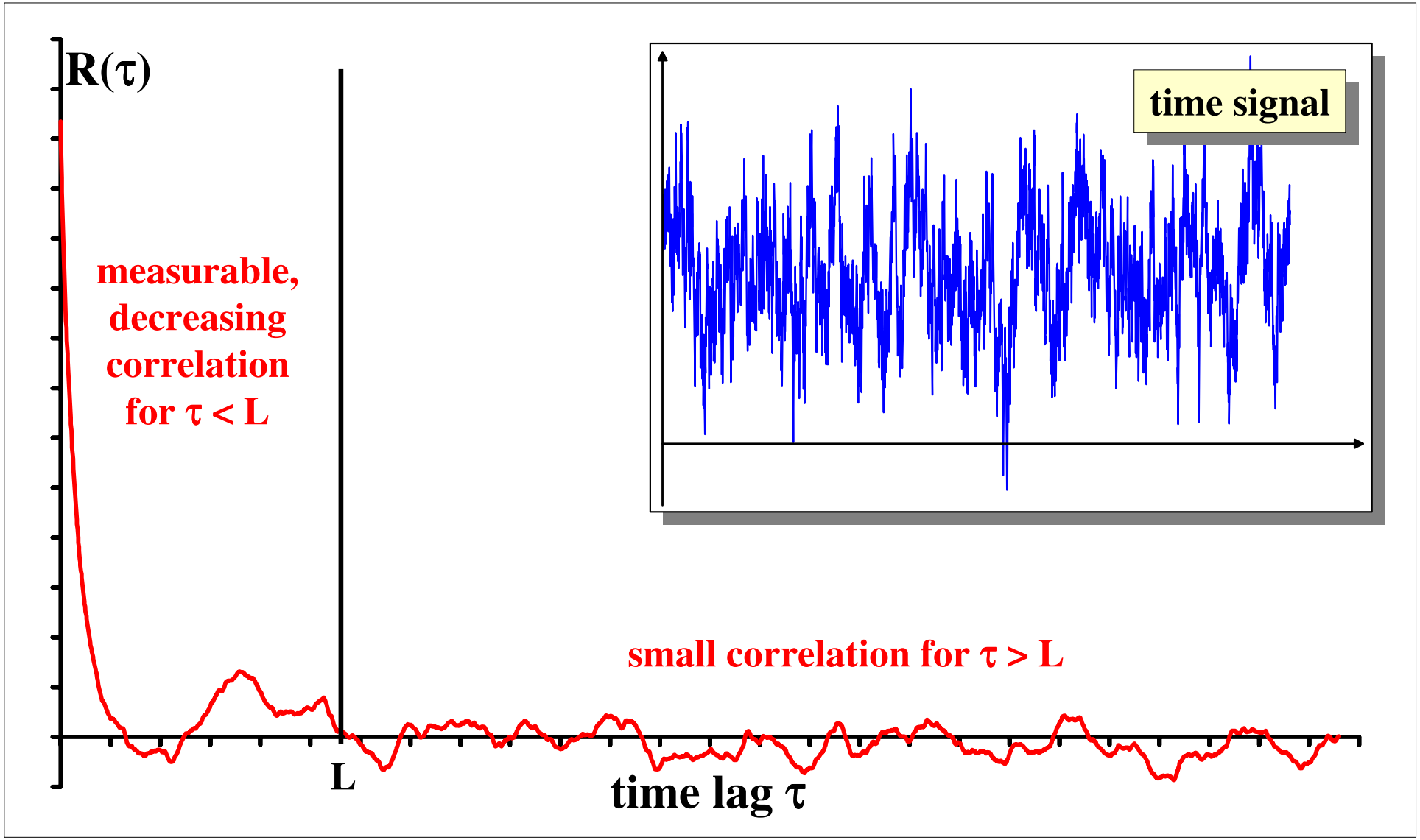
$$R(\tau) = \frac{1}{2} A^2 \cos \Omega \tau$$

$$R(0) = \frac{1}{2} A^2 = \sigma^2$$

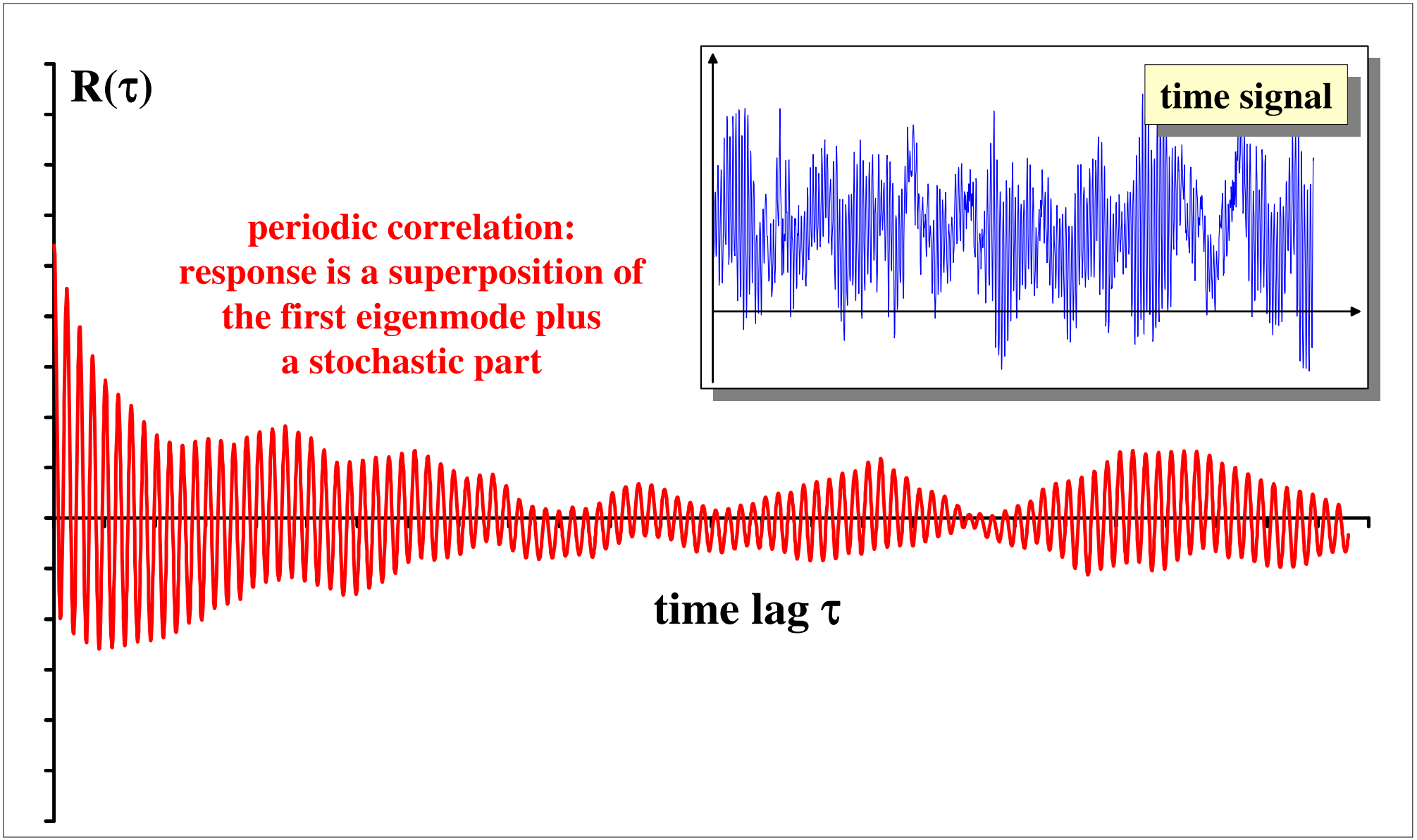
Example 2: White Noise Process



Example 3: History of Wind Speed



Example 4: Response to the Wind Speed



Auto-Covariance Function and Stochastic Process

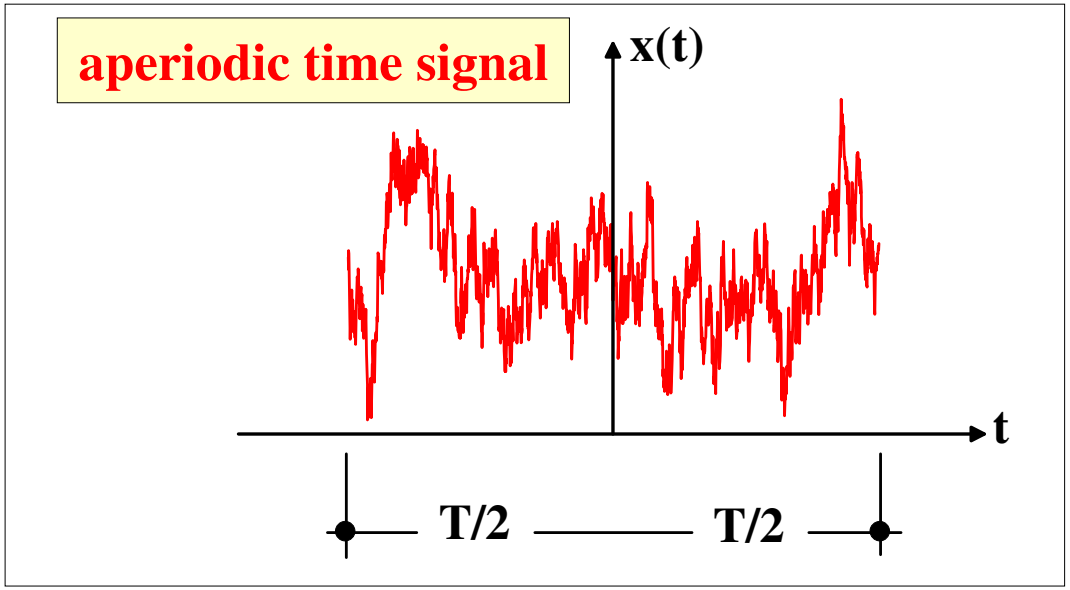
We can experience a *stochastic process* only through its *realizations*. Two realizations which stem from the same process represent the process, and must therefore be *equivalent*. They are *not identical* in the sense that their time functions are identical, but their stochastic – and therefore also dynamic – contents must be identical, except for numerical errors caused by the finite length of the sample.

A stochastic excitation process as a whole induces a vibration in the structure which is also a stochastic process. Again it can only be experienced via its realizations. Now we take two realizations from a given load process and compute the structural responses. We get two results which appear different when we look at the time histories, yet they must be identical with regard to their stochastic and dynamic contents since they are caused by the same underlying excitation process.

That raises the question of when two time histories are stochastically identical, i.e. how can we determine the stochastic process underlying the two processes. We have seen that the statistic parameters are unsuitable: we can easily find many processes which are stochastically distinct but which still have the same standard deviation. The parameter which describes the process unequivocally is the auto-covariance function. Two processes with the same auto-covariance function are identical! That means that all realizations from a given process, no matter how different their time histories may look, would produce, except for numerical errors, the same auto-covariance function. Or vice versa: two time histories with different auto-covariance functions do not stem from the same stochastic process.

Even though the auto-covariance function describes the process fully, it is usually not used in engineering practice. More suitable for practical application is a description in frequency space. That leads to the definition of the *auto-spectrum*. Time domain and spectral domain are generally related by the FOURIER transformation, so we expect also a FOURIER relationship between the auto-covariance function and the auto spectrum.

Stochastic Properties in the FD



Auto-covariance function:

$$R(\tau) = \lim_{T/2 \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x(t+\tau)\, dt$$

Flashback: FOURIER transformation

FD ⇒ TD

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}(\Omega)\, e^{i\Omega t}\, d\Omega$$

TD ⇒ FD

$$\underline{X}(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\Omega t}\, dt \qquad \tilde{\underline{X}}(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{i\Omega t}\, dt$$

We express the function x(t+τ) by a FOURIER integral:

$$x(t+\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}(\Omega) e^{i\Omega t} e^{i\Omega \tau} d\Omega \quad \Rightarrow \quad R(\tau) = \lim_{T/2\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{X}(\Omega) e^{i\Omega t} e^{i\Omega \tau} d\Omega\right\} dt$$

We re-order the sequence of integration:

$$R(\tau) = \lim_{T/2\to\infty} \frac{1}{2\pi T}\int_{-\infty}^{\infty} \underline{X}(\Omega)\left\{\int_{-T/2}^{T/2} x(t)\, e^{i\Omega t} dt\right\} e^{i\Omega \tau} d\Omega$$

We interpret the integral over t as a FOURIER integral and substitute it in the expression of R(τ).

$$\lim_{T/2\to\infty} \int_{-T/2}^{T/2} x(t)\, e^{i\Omega t}\, dt = \int_{-\infty}^{\infty} x(t)\, e^{i\Omega t}\, dt = \underline{\tilde{X}}(\Omega)$$

$$R(\tau) = \frac{1}{2\pi} \lim_{T/2\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \underline{X}(\Omega)\, \underline{\tilde{X}}(\Omega)\, e^{i\Omega\tau} d\Omega = \frac{1}{2\pi} \lim_{T/2\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \left|\underline{X}(\Omega)\right|^2 e^{i\Omega\tau}\, d\Omega$$

The limes is independent of Ω and can be transferred into the integral. It concerns the length of the time signal x(t).

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T/2\to\infty} \frac{1}{T} \left|\underline{X}(\Omega)\right|^2 e^{i\Omega\tau}\, d\Omega$$

We rename the function before the exponential term S and call it auto-spectrum. It is a function of the frequency Ω: S = S(Ω).

$$S(\Omega) = \lim_{T/2\to\infty} \frac{1}{T}\left|\underline{X}(\Omega)\right|^2 \quad \Rightarrow \quad R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\Omega)\, e^{i\Omega\tau}\, d\Omega$$

The auto-covariance function is the integral of the auto-spectrum over the entire frequency domain. Formally it is a FOURIER integral and we see that the auto-spectrum is the frequency domain representation of the auto-covariance function and vice versa.

$$S(\Omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\Omega\tau}\, d\tau = \lim_{T/2\to\infty} \frac{1}{T}\left|\underline{X}(\Omega)\right|^2$$

S(Ω): auto-spectrum of the process, frequency domain representation of the auto-covariance function. We can compute it in two different ways:

- **calculate R(τ), FT yields S(Ω)**
- **FT of x(t), $\underline{X}$(Ω) yields S(Ω)**

Interpretation of the Auto-Spectrum

$$R(\tau = 0) = \sigma^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\Omega)\, d\Omega$$

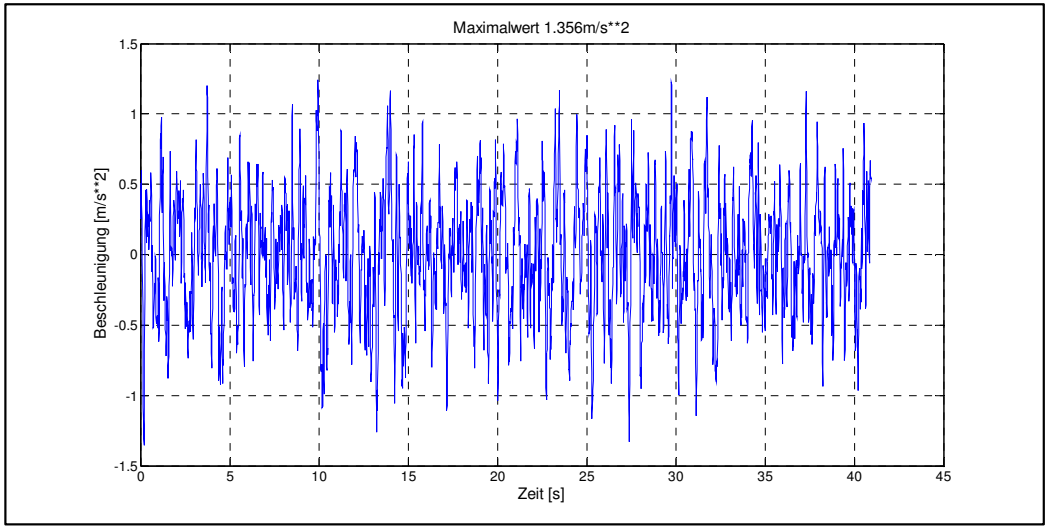
The auto-spectrum is equivalent to the decomposition of the variance into its frequency components. It no longer represents a static view of the dynamic contents of the signal. Since σ is responsible for the dynamic effects, i.e. the vibration energy, we can deduce from the auto-spectrum which frequency ranges contain the highest energy contents. If the maxima of the spectrum of the excitation coincide with the maxima of the transfer function (dynamic amplification) we have the greatest resonance effects. Each contribution $\sigma^2(\Omega)$ to the total variance σ^2 is amplified with the dynamic amplification belonging to that frequency.

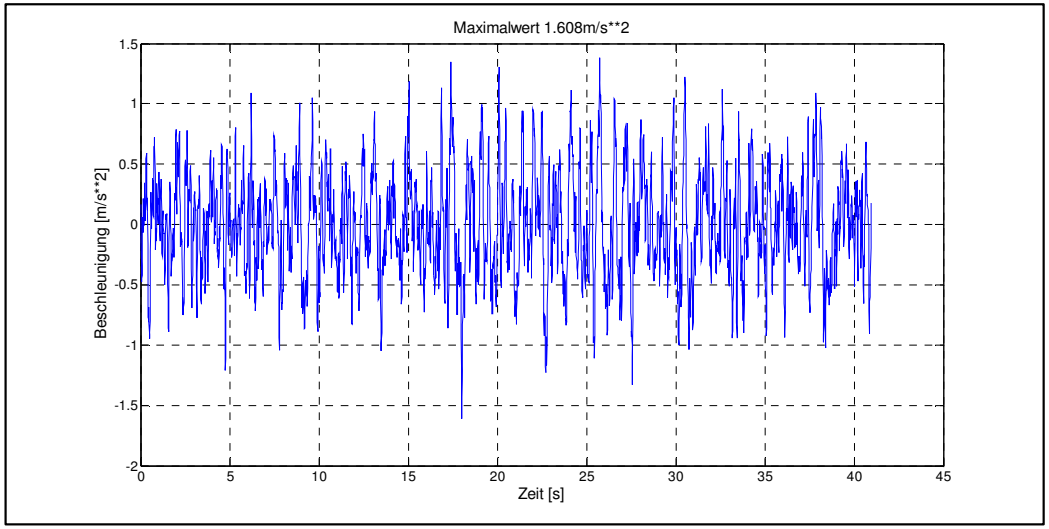
Full name for the auto-spectrum:

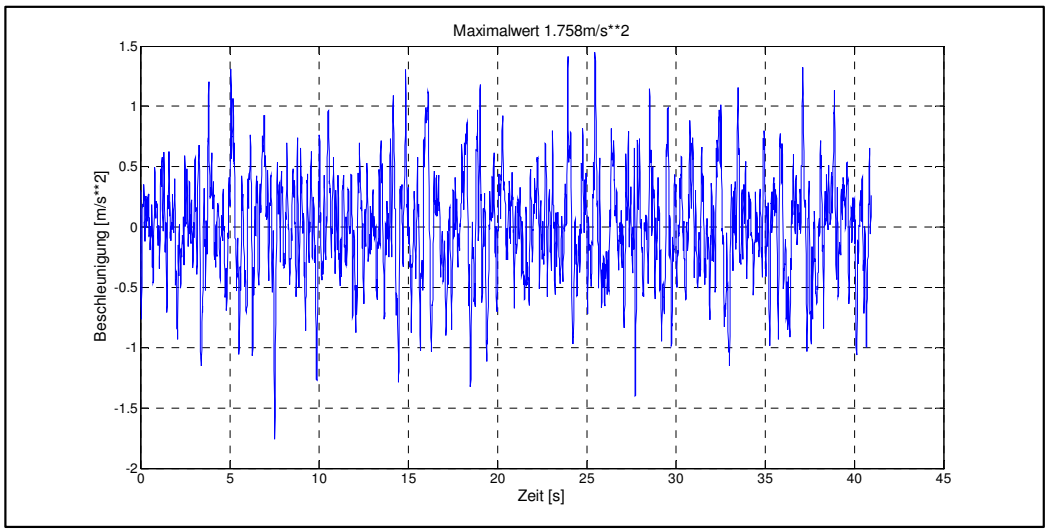
Power Spectral Density Function PSDF

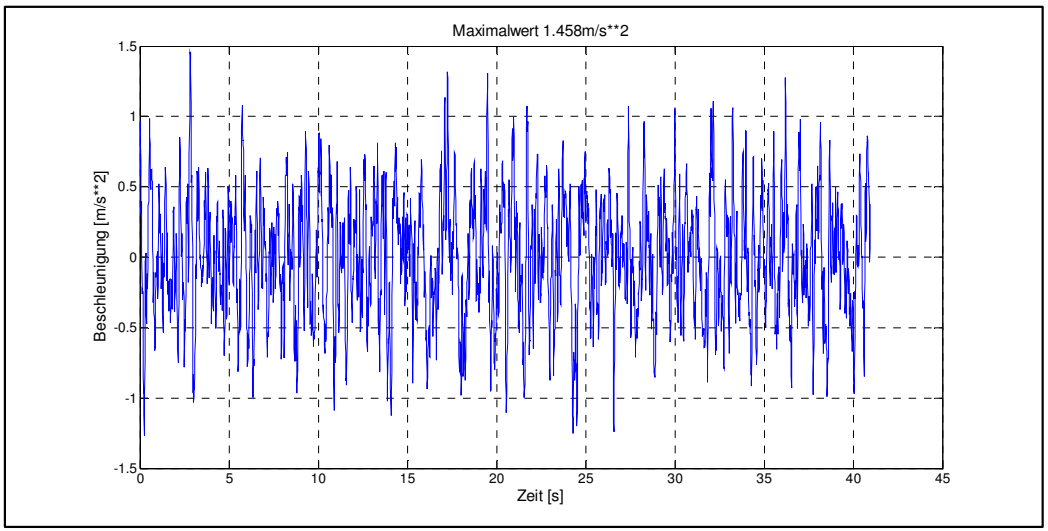
Example: Ground Motion (Acceleration)

Given: 4 acceleration histories with different maximum values (1.356, 1.608, 1.758. 1.458). Question: do they stem from the same stochastic process, i.e. are they stochastically equivalent?



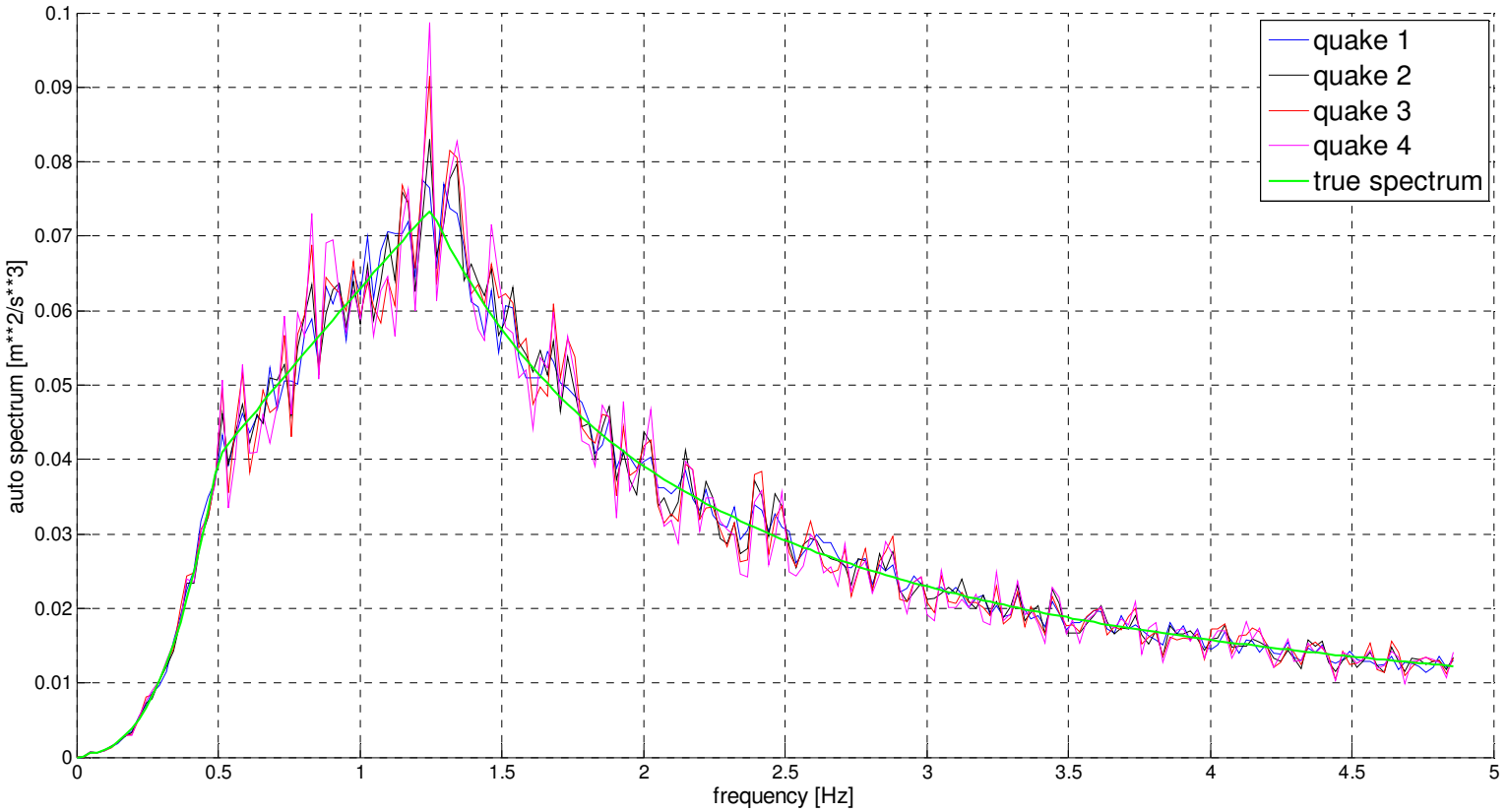






Auto-Spectra of the Quakes

The auto-spectra fluctuate about a *common mean curve*, i.e. they are all realizations of a *common stochastic process*. The mean curve would represent the auto-spectrum of the underlying stochastic process. Since a stationary stochastic process is fully characterized by its auto-spectrum, we can say that the 4 histories are stochastically equivalent.



Attention: Other Form for the PSDF

(a) $$R(\tau) = \int_{-\infty}^{\infty} \hat{S}(\Omega)\, e^{i\Omega\tau}\, d\Omega$$

two-sided, without factor 2π

(b) $$R(\tau) = \int_{0}^{\infty} S^{*}(\Omega)\, e^{i\Omega\tau}\, d\Omega$$

one-sided, without factor 2π

When using spectra from literature, one must make sure of the definition on which the spectra are based!

WIENER-KHINTCHINE Equations

definition:

$$S(\Omega) = \int_{-\infty}^{\infty} R(t)\,(\cos \Omega t - i \sin \Omega t)\,dt$$

The auto-covariance is symmetric: $R(-t) = R(t)$

The integral of a symmetric with an anti-symmetric function is zero

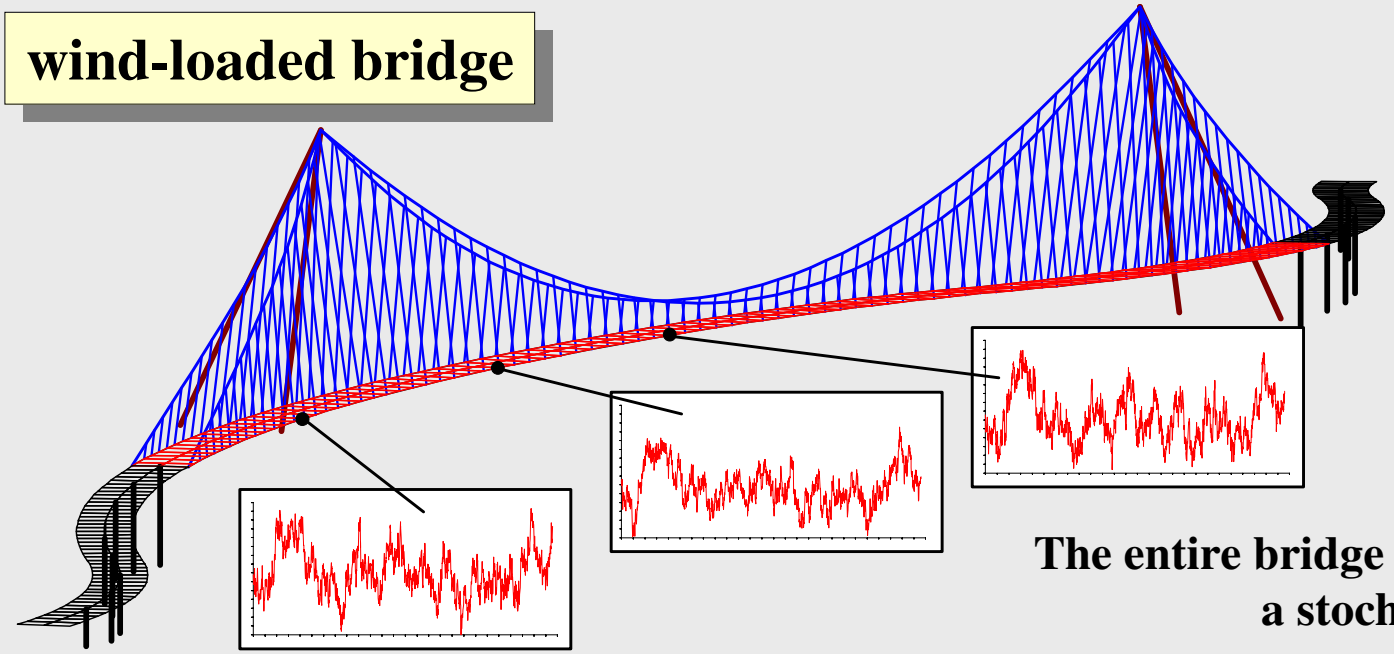
TD ⇒ FD

$$S(\Omega) = 2\int_{0}^{\infty} R(t) \cos \omega t \, dt$$

FD ⇒ TD

$$R(t) = \frac{1}{\pi}\int_{0}^{\infty} S(\Omega) \cos \Omega \tau \, d\Omega$$

Extension to Fields of Processes



The entire bridge is loaded by wind forces: a stochastic process.

At each point the wind speed has a different history: many realisations.

The individual realisations are *independent* yet *correlated*:

- **the histories at neighbouring points are "similar",**
- **at points far apart "different".**

This fact strongly influences the magnitude of the response.

Cross-Covariance of Two Processes

given: 2 processes x_i and x_j:

cross-covariance function $R_{ij}(\tau)$:

$$R_{ij}(\tau) = R_{ji}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x_i(t)\, x_j(t+\tau)\, dt$$

The cross-covariance describes the correlation between two processes. The correlation is not constant with respect to time: two processes might be strongly correlated at the beginning but might lose their correlation as time progresses. The cross-covariance function is therefore formulated as a function of the time lag τ: $R_{ij} = R_{ij}(\tau)$.

Cross-Covariance of a Field of Processes

We can extend the concept of cross-covariance to a field of n random processes x_1, x_2, ..., x_n. Then we can compute a cross-covariance function for each possible pair and gather them into the *cross-covariance matrix*. The diagonal elements are the auto-covariance functions.

cross-covariance matrix R(τ):

$$\mathbf{R}(\tau) = \begin{bmatrix} R_{11}(\tau) & R_{12}(\tau) & \bullet\bullet\bullet & R_{1n}(\tau) \\ R_{21}(\tau) & R_{22}(\tau) & \bullet\bullet\bullet & R_{2n}(\tau) \\ \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet & \bullet\bullet\bullet \\ R_{n1}(\tau) & R_{n2}(\tau) & \bullet\bullet\bullet & R_{nn}(\tau) \end{bmatrix}$$

For a fixed time lag τ, R represents the spatial correlation properties of the field of processes at this time.

Cross-Spectra

The cross-spectra are the FD equivalent to the cross-covariance in the TD. They can be computed either by determining first the cross-covariance matrix R and then transforming each element separately by a FOURIER transformation into the spectral domain, or by computing the FOURIER transform for each process and finding S_{ij} as the product of the individual FOURIER transforms.

$$\underline{S}_{ij}(\Omega) = \int_{-\infty}^{\infty} R_{ij}(\tau)\, e^{-i\Omega\tau}\, d\tau = \lim_{T/2\to\infty} \frac{1}{T}\, \underline{\tilde{X}}_i(\Omega) \cdot \underline{X}_j(\Omega)$$

$$R_{ij}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{S}_{ij}(\Omega)\, e^{i\Omega\tau}\, d\Omega$$

- **The cross-spectra of all processes make up the *cross-spectral matrix*.**
- **The *auto-spectra* are *real*, while the *cross-spectra* are *complex*. The cross-spectral matrix is conjugate complex symmetric.**

Outlook to Structural Design

Given:

(a) Structural design: masses M, stiffness K, damping C.

(b) Loading in the spectral domain: cross-spectral matrix S.

Unknown:

Characteristic values of the structural response which can be compared to permissible values.

Question:

How do we calculate these from the spectral loading data?