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Lecture Series:

Structural Dynamics

Lecture 11:

Earthquake Excitation:

Part B: Response Spectrum Method

Overview

- **Definition of a response spectrum**
  - **Response of an SDOF-system under seismic loading**
  - **Pseudo-spectra**
- **Structural analysis**
  - **Concept**
  - **SRSS superposition**
- **Design response spectra (EUROCODE 8)**
- **Example**

Response of an SDOF-System to Seismic Loading

Response spectra result from the response of a single-degree-of-freedom system to seismic excitation. We start by looking at the equation of motion:

$$m\ddot{v} + c\dot{v} + kv = -m\ddot{v}_g \quad \Rightarrow \quad m\ddot{v} + 2\xi m\omega\dot{v} + kv = -m\ddot{v}_g$$

$$\ddot{v} + 2\xi\omega\dot{v} + \omega^2 v = -\ddot{v}_g$$

The seismic load is proportional to the mass. This has the effect that the equation of motion does not, after division by m, depend on the actual values of k and m, but only on the eigenfrequency of the system. All SDOF-systems with the same eigenfrequency show exactly the same response to a given acceleration history $a_g(t)$! Therefore it is possible to calculate the response for all possible SDOF-systems to a given $a_g(t)$ beforehand by looping over all eigenfrequencies. We plot the maximum response, i.e. the maximum displacement, velocity and acceleration versus the period (not the frequency) of the SDOF-system. Such a plot contains full information regarding the maximum response for the quake in question. Such a function is called *response spectrum*.

Generation of a Response Spectrum

choose accelerogram → choose damping ξ → loop over the frequency domain: TD analysis of SDOF-systems

↓

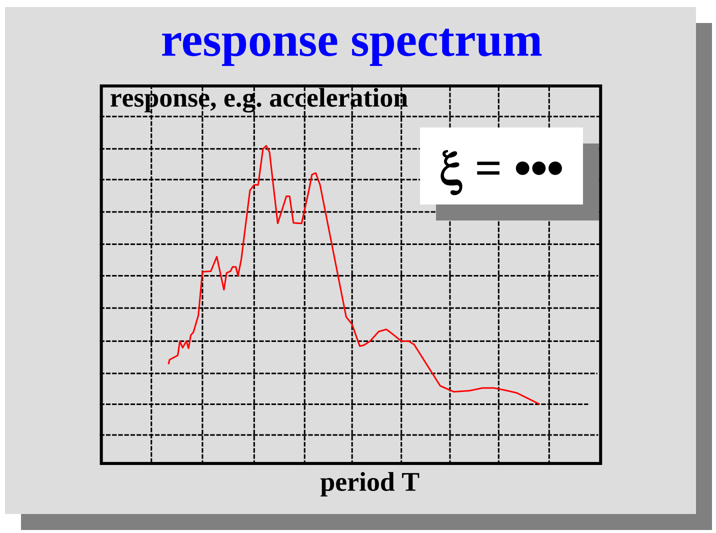
extract maximum values

↓

plot over period T

different earthquake

different damping



Types of Response Spectra

The spectral ordinates are generally denoted by S. We have:

- **The displacement spectrum** S_d **for the** ***relative displacement*****.**
- **The velocity spectrum** S_v **for the** ***relative velocity*****.**
- **The acceleration spectrum** S_a **for the** ***absolute acceleration*****.**

If the SDOF-oscillator responds ***harmonically*** **(as in the case of a pure white noise process), and the damping is zero, we have the relationship:**

$$S_a = \omega S_v = \omega^2 S_d$$

If that is not the case, and we still derive S_a and S_v from S_d via ω, we talk of "*pseudo-spectra*". We will se later that EUROCODE 8 only defines acceleration spectra, so that if we need velocity or displacement spectra we have to derive them as pseudo-spectra.

Structural Analysis

We now think back to the mode superposition method. Its basic idea was to decouple the equation of motion into a number of independent SDOF-oscillators. Each oscillator can then be computed separately, and the solution of the original structure can be obtained by a superposition of the SDOF-solutions. The attractiveness of the method lies in the physical property that higher mode shapes contribute less to the total response than lower ones, so that we could neglect all modes above some critical threshold.

A structural analysis with the response spectrum method is similar. Again we transform the problem into modal space and again we can reduce the number of relevant modal dofs. In modal space we "calculate" the maximum response by a response spectrum, i.e. we simply take the appropriate value from the response spectrum calculated beforehand. The entire "computation" consists in a simple lookup of a scalar value. It is extremely simple.

The simplicity, however, is bought at a certain price. The response spectra yield only the *maximum absolute values*, not the time instances at which these occur. We have looked up N_{mode} maximum values for the N_{mode} modes, but it would be wrong to simply add up these maxima since they occur at different times. We need special superposition techniques to account for this problem.

Before we come to the superposition concepts, we take a closer look at the decomposition into modal space for response spectra.

Transformation into Modal Space

Modal decomposition was treated in Dynamics I, Lecture 7. We start from the coupled system of differential equations in the *finite element solution space*:

$$\mathbf{M}\,\ddot{\mathbf{V}}_{\text{rel}} + \mathbf{C}\,\dot{\mathbf{V}}_{\text{rel}} + \mathbf{K}\,\mathbf{V}_{\text{rel}} = \mathbf{P}_{\text{quake}} = -\mathbf{M}\,\ddot{\mathbf{V}}_{\text{ground}} = -\mathbf{M}\,\mathbf{X}\,a_g(t)$$

We dispense with the index $()_{\text{rel}}$ for simplicity's sake. Now we define *modal degrees of freedom* η(t) which we relate to the vector of nodal degrees of freedom V(t) via

$$\mathbf{V}(t) = \mathbf{\Phi}\cdot\eta(t) = \mathbf{\Phi}_1\cdot\eta_1(t) + \mathbf{\Phi}_2\cdot\eta_2(t) + \mathbf{\Phi}_3\cdot\eta_3(t) + \ldots$$

In modal space the system becomes uncoupled:

$$\tilde{m}_i\ddot{\eta}_i + 2\xi_i\tilde{m}_i\omega_i\dot{\eta}_i + \tilde{k}_i\eta = \tilde{p}_i$$

modal damping!

$$\tilde{m}_i = \mathbf{\Phi}_i^T\cdot\mathbf{M}\cdot\mathbf{\Phi}_i \qquad \tilde{k}_i = \mathbf{\Phi}_i^T\cdot\mathbf{K}\cdot\mathbf{\Phi}_i$$

$$\tilde{p}_i = -\mathbf{\Phi}_i^T\cdot\mathbf{M}\cdot\mathbf{X}\cdot a_g$$

Modal Responses

We can perform the time-independent matrix operations before we start with the time domain solution:

$$\tilde{p}_i = -\boldsymbol{\Phi}_i^T \cdot \mathbf{M} \cdot \mathbf{X} \cdot a_g = -\tilde{\beta}_i a_g(t)$$

The factors β_i with the tilde represent *modal seismic masses*. They lead to *modal inertial forces* which produce the *true modal response η_{mod}*.

We have proven earlier in this lecture that the response of an SDOF-system under seismic loads does not depend on the values for mass and stiffness, but only on its period. This, however, is only true if the seismic mass on the right-hand side is identical to the *structural mass* of the oscillator.

Then, and only then, the mass cancels out and we can *look up* the maximum response in a *response spectrum*. We denote this look-up solution with η_{resp}. Our modal oscillator, however, does not meet this requirement. The modal structural mass is computed by multiplying the mass matrix from both left and right with the mode shape, while we perform only one single multiplication for the modal seismic masses. So we cannot look up the maximum modal response without further deliberations.

Scaling of the Modal Seismic Masses

We introduce a dimensionless *scaling factor α_i* in each modal equation such that the *scaled modal seismic* masses become identical to the *modal structural masses*:

$$\alpha_i \tilde{\beta}_i = \tilde{m}_i \quad \Rightarrow \quad \tilde{\beta}_i = \frac{1}{\alpha_i}\tilde{m}_i \quad \Rightarrow \quad \frac{1}{\alpha_i} = \frac{\tilde{\beta}_i}{\tilde{m}_i}$$

The scaling factor changes the differential equation to:

$$\tilde{m}_i \ddot{\eta}_i + 2\xi_i \tilde{m}_i \omega_i \dot{\eta}_i + \tilde{k}_i \eta = -\tilde{\beta}_i a_g = -\frac{1}{\alpha_i}\tilde{m}_i a_g = \frac{1}{\alpha_i}\tilde{p}_{resp}$$

The response-spectrum solution η_{resp} corresponds to the *scaled load p_{resp}*. We can look it up directly without any problems. Since the *true load* is smaller by the factor α_i, we simply have to divide the look-up solution by the factor α_i.

True Modal Response

The true modal response is then given by:

$$\eta_{mod,i} = \frac{1}{\alpha_i}\eta_{resp,i} \quad \textbf{with} \quad \frac{1}{\alpha_i} = \frac{\tilde{\beta}_i}{\tilde{m}_i}$$

Often in books the mode shapes are chosen such that the modal masses becomes unity (i.e. equal to one mass unit – 1 tonne, 1 kg, …). The scaling factors then reduce to

$$\frac{1}{\alpha_i} = \frac{\tilde{\beta}_i}{1\,\text{mass unit}} = \beta_i$$

Note: the beta-factors *without tilde* are *dimensionless scaling factors*, while the beta *with tilde* has the *dimension of mass*. The numerical values are identical, but their units are different.

Maximum Mode-Wise Response

The maximum modal response can be transformed back into the original finite element space:

$$\mathbf{V}(t) = \mathbf{\Phi} \cdot \eta(t)$$

The above equation holds for all time instances. For the maximum responses of the i-th mode we get:

$$\mathbf{V}_{max,i} = \mathbf{\Phi}_i \cdot \eta_{mod,max,i}(t) = \mathbf{\Phi}_i \cdot \beta_i \cdot S_{d,i}$$

$$\dot{\mathbf{V}}_{max,i} = \mathbf{\Phi}_i \cdot \dot{\eta}_{mod,max,i}(t) = \mathbf{\Phi}_i \cdot \beta_i \cdot S_{v,i}$$

$$\ddot{\mathbf{V}}_{max,i} = \mathbf{\Phi}_i \cdot \ddot{\eta}_{mod,max,i}(t) = \mathbf{\Phi}_i \cdot \beta_i \cdot S_{a,i}$$

The mode-wise maximum responses must finally be aggregated to maximum responses for the true structure. This leads us directly to the question of spectral superposition.

Response Spectrum Method

modal decomposition

→

modal space

equivalent static forces: static analysis

max. response from response spectra

displacement-based FE-analysis

→

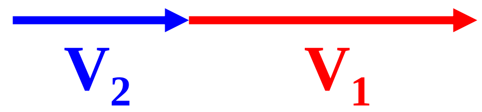
superposition: SRSS-method

The Superposition Problem

Problem: **We know the maximum values of 2 variables which are independent, but we don‘t know the times at which these maxima occur.**

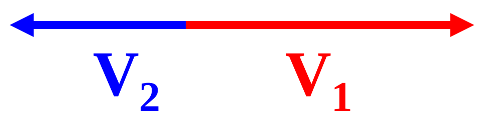
Question: **What is the maximum of the sum of these variables?**

Case 1:
perfect positive correlation



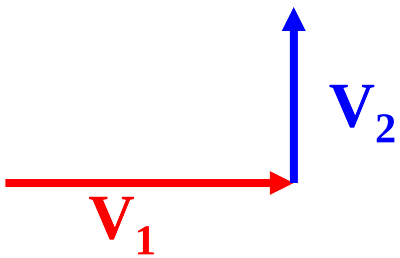
$$|\mathbf{V}| = |\mathbf{V}_1| + |\mathbf{V}_2|$$

Case 2:
perfect negative correlation



$$|\mathbf{V}| = |\mathbf{V}_1| - |\mathbf{V}_2|$$

Case 3:
no correlation



$$|\mathbf{V}| = \sqrt{\mathbf{V}_1^2 + \mathbf{V}_2^2}$$

The SRSS-Method

The ***SRSS-method*** (***Square Root*** of the ***Sum*** of ***Squares***) assumes uncorrelated variables. It reads for the general case of n independent variables:

$$V_{max} = \sqrt{\sum_{i=1}^{Nmod} V_i^2}$$

It gives good results if the eigenfrequencies of the modes are widely spaced apart. The quality is not that good if we have closely spaced eigenfrequencies. For these cases a more advanced method, the ***CQC-method*** (***Complete Quadratic Combination***) which is based on the theory of random vibrations, has been developed.

The CQC-Method: Theory

The CQC method assumes a correlation of all modes, as witnessed in the double sum. The correlation is captured by the *cross-modal coefficients* ρ_{ij}. They depend on the damping ξ and the ratio r of two eigenfrequencies, where the quotient is computed such that r is always less equal one.

$$V_{max} = \sqrt{\sum_{i=1}^{Nmod}\sum_{k=1}^{Nmod} V_i \rho_{ik} V_k}$$

$$\rho_{ij} = \frac{8\xi^2(1+r)r^{3/2}}{(1-r^2)^2 + 4\xi^2 r(1+r)^2}$$

$$r = \frac{\min(\omega_i, \omega_k)}{\max(\omega_i, \omega_k)} \le 1$$

The diagonal terms (r=1) are unity, while the off-diagonal depend on r. For a cluster of closely spaced eigenfrequencies (r≈1) we get rho-values in the vicinity of one, for widely spaced frequencies (r ≈0) the correlation drops to almost zero.

The CQC method then degenerates to the SRSS method. So it is a more general method which contains the classic SRSS as a special case. Most modern programs therefore contain CQC as the more general case.

A numerical example which demonstrates the limits of SRSS can be found in the book by Edward L. Wilson “Three-Dimensional Static and Dynamic Analysis of Structures”. He discusses a four-storey three-dimensional building.

The CQC-Method: Example

Wilson computed the first five eigenfrequencies to:

$$\boldsymbol{\omega} = \begin{bmatrix} 13.869 & 13.931 & 43.995 & 44.189 & 54.418 \end{bmatrix}$$

We observe two clusters of closely spaced frequencies: (ω_1/ω_2) and (ω_3/ω_4). The gap between cluster 1 and cluster 2 is large, the gap between cluster 2 and ω_5 is moderate. For ξ = 5 % we get the following matrix of cross-modal parameters:

$$\boldsymbol{\rho} = \begin{bmatrix} 1.0000 & 0.9991 & 0.0057 & 0.0057 & 0.0037 \\ 0.9991 & 1.0000 & 0.0058 & 0.0057 & 0.0037 \\ 0.0057 & 0.0058 & 1.0000 & 0.9981 & 0.1796 \\ 0.0057 & 0.0057 & 0.9981 & 1.0000 & 0.1859 \\ 0.0037 & 0.0037 & 0.1796 & 0.1859 & 1.0000 \end{bmatrix}$$

We have a tight correlation within the two clusters – shown by the two green sub-matrices. There is no correlation between cluster 1 and the rest, and a moderate correlation of about 18 % between cluster 2 and the last frequency ω_5.

Use of Response Spectra

Response spectra depend on:

• The local situation: ground conditions etc.

• The earthquake.

For each site it is necessary to generate a site-specific spectrum.

Question:

Definition of *design spectra* which can be parametrized to take account of the local conditions?

Design Response Spectra

A *design response spectrum* is a simplified spectrum which is defined in a general way in a code. For the *specific problem* it can be *specified* by free parameters which capture the *local conditions*.

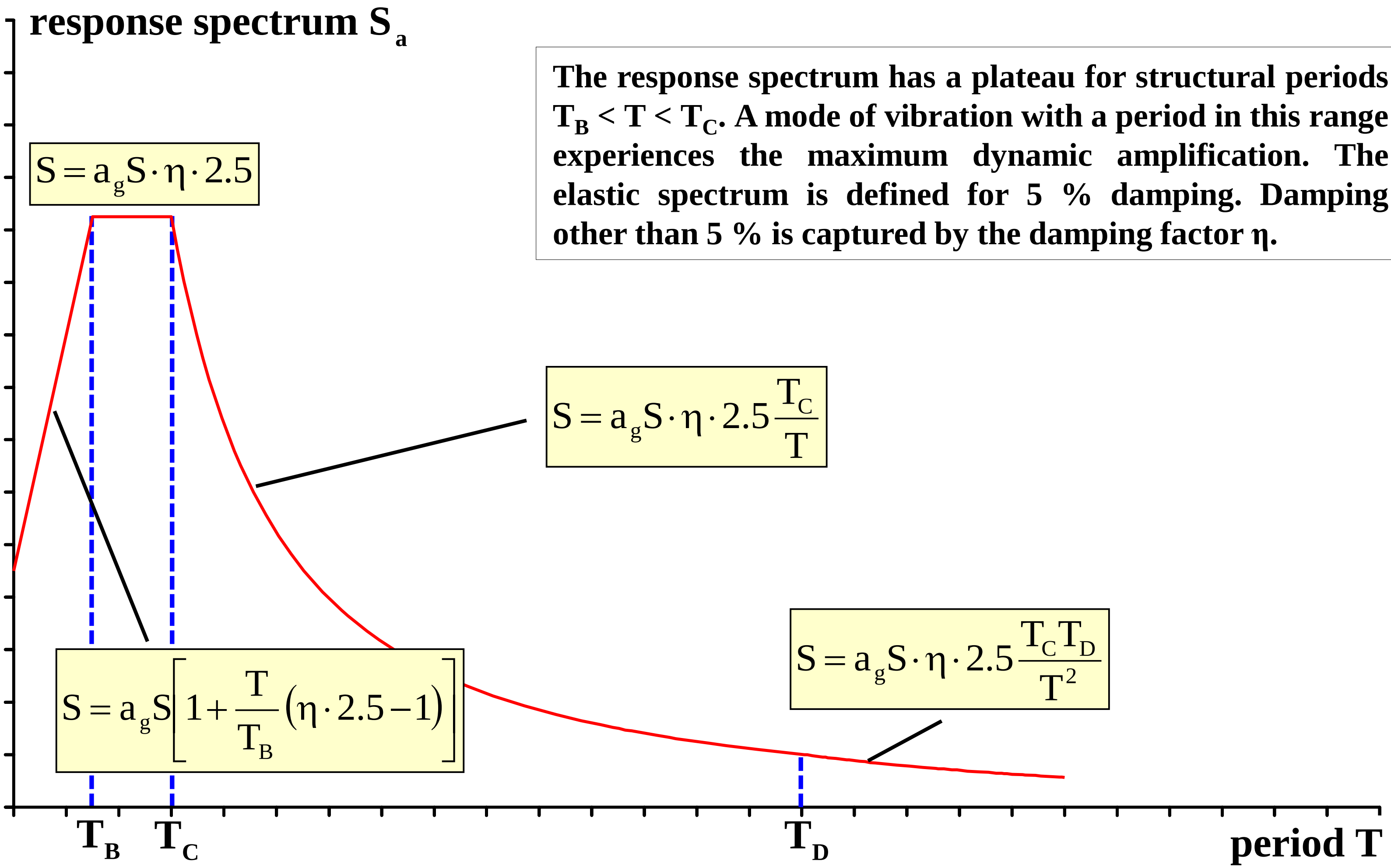
A design spectrum addresses in particular the problem of the statistic reliability of the results. It is defined in such a way that it forms an *envelope* encompassing all possible earthquakes at a given location. Therefore it produces results which are on the safe side and only one *single analysis* is necessary to account for all possible cases.

In Europe seismic design is regulated in *EUROCODE 8* "*Design of structures for earthquake resistance*". Part 1 "*General rules, seismic actions and rules for buildings*" regulates the more general aspects, while Parts 2 through 6 deal with specific types of structures such as bridges (Part 2), tanks, silos and pipelines (Part 4) or towers, masts and chimneys (Part 6).

Design spectra are defined in Part 1, separately for the horizontal and the vertical components. Also there is a differentiation between *elastic response spectra* and *inelastic design spectra*. The parameters defining the spectra are:

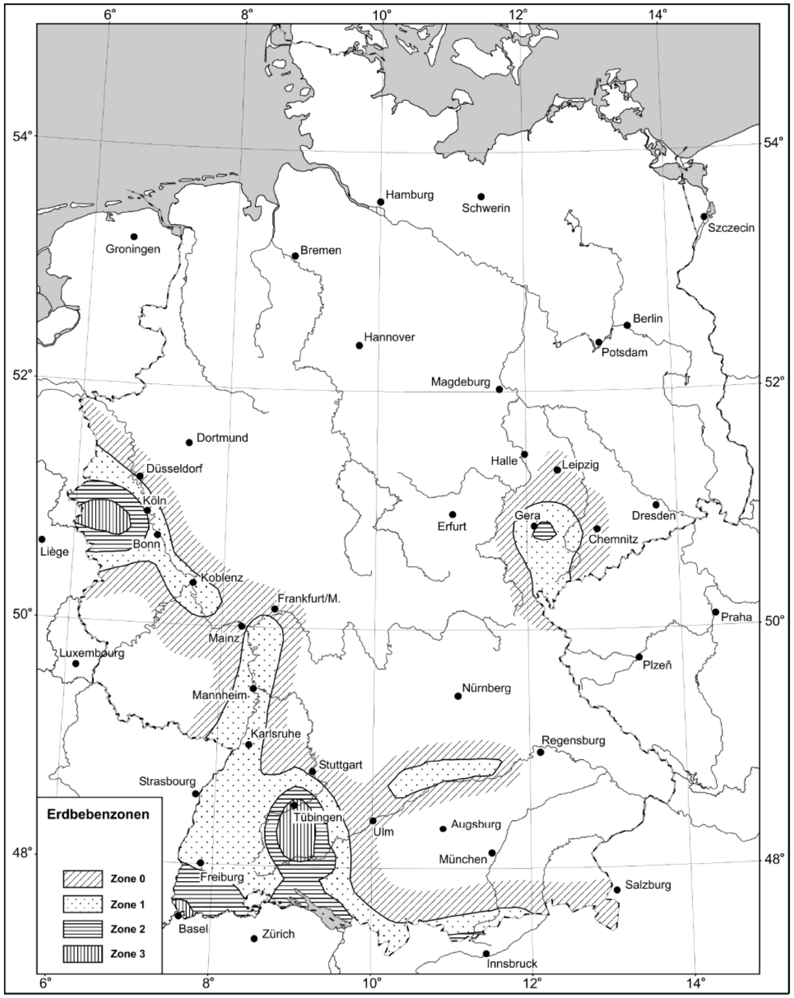
- The global seismic activity – seismic zone.
- The type of subsoil.
- The damping properties.
- The ductility of the structure.

Elastic Spectrum for Horizontal Accelerations



The response spectrum has a plateau for structural periods $T_B < T < T_C$. A mode of vibration with a period in this range experiences the maximum dynamic amplification. The elastic spectrum is defined for 5 % damping. Damping other than 5 % is captured by the damping factor η.

Earthquake Zones for Germany



The parameter a_g describes the design intensity of the quake. It is the value of the ground acceleration at the base rock. Its value can be taken from a seismic map from the National Annex to EC 8. If such a map does not exist, then a_g must be determined from the records of seismic activity at the specific location. The determination of a_g is a task for a seismologist.

ground peak acceleration according to DIN 4149 and EC8				
seismic zone	0	1	2	3
a_g [m/s²]	---	0.4	0.6	0.8

Taken from:

1. DIN 4149: „ Bauten in deutschen Erdbebengebieten – Lastannahmen, Bemessung und Ausführung üblicher Hochbauten“, April 2005.
2. Eurocode 8: „Auslegung von Bauwerken gegen Erdbeben – Teil 1: Grundlagen, Erdbebeneinwirkungen und Regeln für Hochbau, Nationaler Anhang – National festgelegte Parameter“, Januar 2011.

Soil Effects

The earthquake experiences a modification by passing from the base rock to the surface through the subsoil. These modifications concern both a possible amplification and a shift of the spectral content of the acceleration history. The amplification is captured by the *soil factor S* and the shift of the spectral content by changing *time boundaries T_B, T_C, T_D*. EC 8 contains tables for different earthquake types for the standard soils types A – D.

elastic response spectrum type 1, horizontal component				
soil type	S	T_B [s]	T_C [s]	T_D [s]
A	1.00	0.15	0.40	2.0
B	1.20	0.15	0.50	2.0
C	1.15	0.20	0.60	2.0
D	1.35	0.20	0.80	2.0
E	1.40	0.15	0.50	2.0

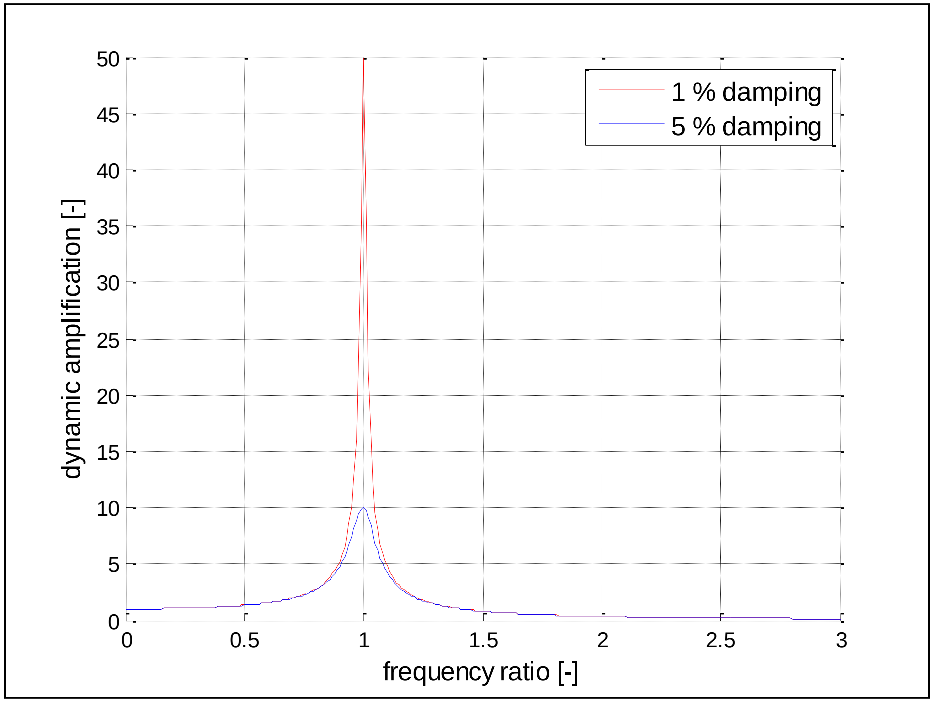
Soil Types

EC 8 defines five standard types of soil for which the parameters S, T_B, T_C, T_D are given. If the local soil situation cannot be captured by these standard classes, then special seismological simulations must be performed to determine these above parameters.

Typ	Beschreibung
A	Fels oder andere felsähnliche geologische Formation, mit höchstens 5 m weicherem Material an der Oberfläche.
B	Ablagerungen von sehr dichtem Sand, Kies oder sehr steifem Ton, mit einer Dicke von mindestens einigen zehn Metern, gekennzeichnet durch einen allmählichen Anstieg der mechanischen Eigenschaften mit der Tiefe.
C	Tiefe Ablagerungen von dichtem oder mitteldichtem Sand, Kies oder steifem Ton, mit Dicken von einigen zehn bis mehreren hundert Metern.
D	Ablagerungen von lockerem bis mitteldichtem kohäsionslosem Boden (mit oder ohne einige weiche kohäsive Schichten), oder von vorwiegend weichem bis steifem kohäsivem Boden.
E	Ein Bodenprofil, bestehend aus einer Oberflächen-Alluvialschicht mit vs-Werten nach C oder D und veränderlicher Dicke zwischen etwa 5 m und 20 m über steiferem Bodenmaterial mit v_s > 800 m/s.

Damping

The response spectra in EC 8 have all been computed by for a damping of 5 %. Theoretically, it would be necessary to re-compute the response spectra for damping ratios different from 5 % since the damping influences different harmonics in different ways: there is a pronounced dependency of the dynamic amplification in the vicinity of the resonance point which vanishes when moving away from it. To simplify things, EC 8 provides a ***damping correction factor η*** to account for differing damping ratios.



damping parameter for $\xi \neq 5$ %

$$\eta = \sqrt{\frac{7}{2+100\xi}} \geq 0.7$$

Ductility

Seismic effects within a structure depend not only on the intensity and spectral distribution of the quake, but also on the *ductility of the structure*. We have no external loads which must be equilibrated by the inner forces. Instead the ground moves, and the seismic forces are created by the inertia of the structure *which tries to follow the ground motion since it is connected to the ground*. If, in the extreme case, structure and ground we uncoupled, e.g. if the building rested on a smooth sliding surface, then the ground could move independently from the structure above it and we would have no seismic excitation. The ground would move, and the structure would remain at rest with regard to a fixed outside observer.

A similar, yet not so extreme effect occurs when we have inelastic ductile deformations. Inelastic behaviour, e.g. cracking of concrete or plastification of the reinforcement, reduces the member stiffness of the afflicted members. Instead of transmitting the ground motion into the structure, the members deform and we have a partial decoupling of the ground motion from the motion of the building.

These effects are nonlinear in nature and could be captured by a nonlinear time domain analysis. Our computational model would have to include nonlinear constitutive laws for our building materials – steel and reinforced concrete – and we could update the material status at each time step, thus taking account of the damage evolution in our structure. Typically, damage would first develop locally in some critical members and large deformations develop also locally. The seismic effects within the remaining undamaged structure are greatly reduced. Such simulations are feasible, yet they are extremely difficult to perform – profound knowledge of nonlinear computational methods is required – and also computationally time consuming. Once again we would have the problem that one analysis is not sufficient to capture the stochastic process per se. So we had to perform a batch of analyses which increases the effort further.

To avoid these too complex computations, EC 8 provides a simplified way to account for ductile behaviour.

Inelastic Design Spectra

We have seen that ductility reduces seismic effects. Not because the quake itself is smaller – the quake has no knowledge of the structural ductility – but because the inertial forces are reduced. The response spectrum method is a linear method: we reduce the dynamic problem to a static one where the dynamic effects are captured in the values of S_a, i.e. the inertial loads. The SDOF-oscillator is per force physically linear, so it is not possible to calculate physically nonlinear response spectra which model the damage evolution in time as does a time domain analysis.

Instead EC 8 provides the *ductility factor q* which is an overall measure of the ductility of the entire structure. The response spectrum (c.f. next page) is basically divided by q so that a q of 3 means that the inner forces are reduced to one third.

ductility factor q			
	rigid buildings	"normal" buildings	ductile buildings
q	1	2 ÷ 3	3 ÷ 5

Ductile Design

We can only reduce the inner forces if we make sure that ductility effects can in fact be activated. Two requirements must be met:

Inelastic deformations must be possible. A bridge pylon, e.g., is subjected to high static pressure due to the dead weight of the bridge. The vertical acceleration is usually not sufficiently large to overcome the static pressure so that no tension cracking occurs. As a consequence there is no stiffness reduction and q must be set to 1. We have bending, however, for the horizontal acceleration component, tension cracking can occur and we can design the pylon such that q is larger than 1.

Inelastic deformation must be sustainable. The inelastic deformations must not lead to a complete destruction of the member. Even though the member experiences damage in such a way that its stiffness is reduced and large displacements occur, the member must retain its structural integrity in its damaged state. We must ensure this condition by special calculations such a the so-called *capacity design*. In a capacity design we check the *existing ductility capacity* against the *required ductility* which depends on the q-factor we want to attain. If the existing ductility is too small, we must redesign the member such that its ductility is increased to the required level. Capacity design is not a topic we pursue here further; it is part of the theory of concrete structures.

Horizontal Response Spectra

Just as an example we see here the formulas for both the elastic and the design spectra from EC 8.

Response spectra for the horizontal acceleration component			
No	region	elastic spectrum	inelastic design spectrum
1	$0 < T \le T_B$	$S_e(T) = a_g S\left[1 + \frac{T}{T_B}(\eta \cdot 2.5 - 1)\right]$	$S_d(T) = a_g S\left[\frac{2}{3} + \frac{T}{T_B}\left(2.5\frac{1}{q} - \frac{2}{3}\right)\right]$
2	$T_B \le T \le T_C$	$S_e(T) = a_g S \cdot \eta \cdot 2.5$	$S_d(T) = a_g S \cdot 2.5\frac{1}{q}$
3	$T_C \le T \le T_D$	$S_e(T) = a_g S \cdot \eta \cdot 2.5\frac{T_C}{T}$	$S_d(T) = a_g S \cdot 2.5\frac{1}{q}\frac{T_C}{T}$
4	$T_D \le T$	$S_e(T) = a_g S \cdot \eta \cdot 2.5\frac{T_C T_D}{T^2}$	$S_d(T) = a_g S \cdot 2.5\frac{1}{q}\frac{T_C T_D}{T^2}$

EC 8 – Summary

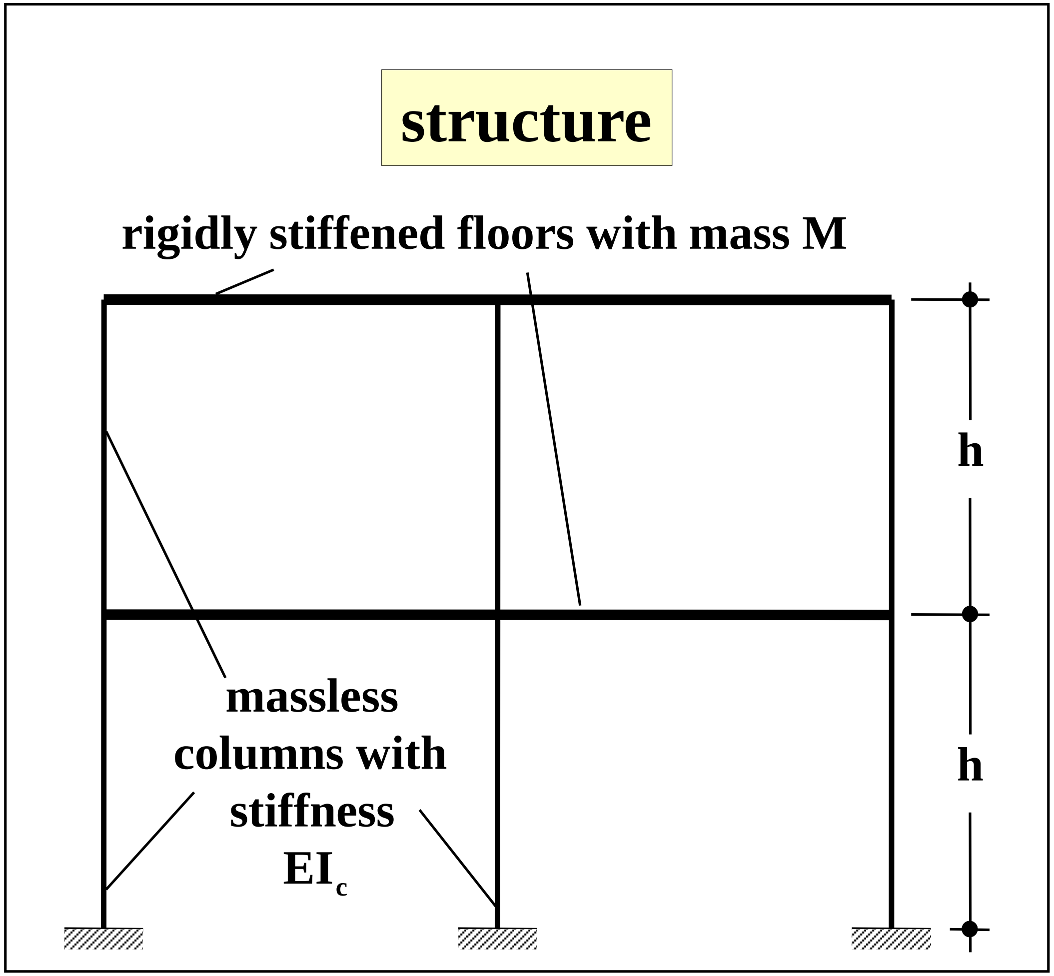
EUROCODE 8 regulates the "design of structures for earthquake resistance". It is a substantial work with hundreds of pages which addresses a multitude of aspects. In this lecture we could only highlight very few of them and give a rough impression.

The practical application of EC 8 is not trivial. It should be studied carefully before producing a practical design. Also it should be mentioned that EC 8 contains several inconsistencies which can only be overcome by interpreting the code, i.e. a literal application is not always possible.

One example is an inconsistency between the elastic spectra and the design spectra. If we perform a TD analysis, we would have to use ***acceleration histories*** which are compatible with the ***elastic 5%-spectrum***. Damping and ductility effects are then part or our computational model. That means that the elastic 5%-spectrum, for $\xi = 5$ % and q = 1, serves directly as our ***design spectrum***. That entails that the design spectrum should become identical to the elastic spectrum for $\xi = 5$ % and q = 1. That, however, is not the case for the period range $0 < T < T_B$. So we have an inconsistent situation for the period range $0 < T < T_B$, where the TD simulation is not equivalent to the response spectrum solution.

So EC 8 should be applied carefully with a critical eye to other possible inconsistencies.

Example: Plane Frame



We study a typical two-story building whose load bearing structure is made up of an elastic frame. The mass is mainly concentrated in the floors which we treat as being rigid.

data

$H = 4.0$ m

$M = 40$ tons

$EI_c = 16000$ kNm^2

$\xi = 5$ %

Step 1: Discretized Model

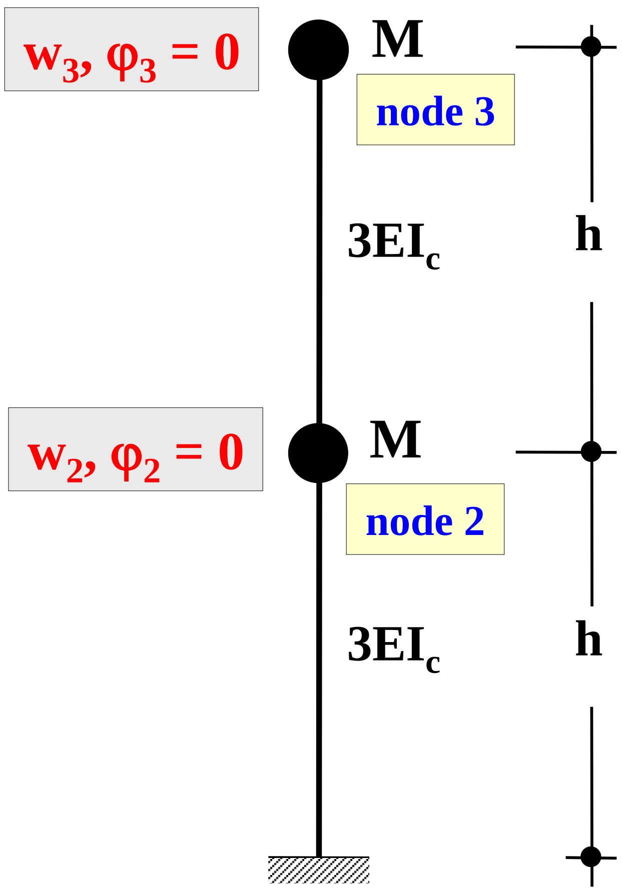
We build our numerical model with the ***direct stiffness method***. Here we introduce one central simplification to reduce the number of kinematic unknowns: we assume that the floors are rigid. This assumption is justified if we compare the extensional stiffness of a solid floor with the stiffness of the columns which are several meters space apart.

⇒ (a) the rotations are zero.

⇒ (b) the horizontal displacements within each floor are identical.

From (b) it follows that the stiffness of the three columns in each story can be combined into an equivalent column with $3EI_c$. Thus our numerical model has only ***two degrees of freedom***: the two floor displacements w_2 and w_3.

model with 2 dofs



Element Stiffness Matrices

generic 2D beam (dofs w_i, φ_i, w_k, φ_k):

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}$$

element 1: dof w_2

$$\mathbf{k}_1 = \frac{3EI_c}{h^3}\left[12\right]$$

element 2: dofs w_2, w_3

$$\mathbf{k}_2 = \frac{3EI_c}{h^3}\begin{bmatrix} 12 & -12 \\ -12 & 12 \end{bmatrix}$$

System Matrices

stiffness matrix

$$\mathbf{K} = \frac{3EI_c}{h^3}\begin{bmatrix} 24 & -12 \\ -12 & 12 \end{bmatrix} = \begin{bmatrix} 18000 & -9000 \\ -9000 & 9000 \end{bmatrix}$$

mass matrix

$$\mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix}$$

Step 2: Eigenfrequency Analysis

The first analysis always consists in an ***eigenmode analysis***. Our purpose is twofold: we need the modal parameters for the transformation into modal space and we want to develop a feeling for the modes of vibration.

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

$$\begin{vmatrix} 18000 - 40\omega^2 & -9000 \\ -9000 & 9000 - 40\omega^2 \end{vmatrix} = 0$$

$\omega_1 = 9.27$ rad/s

$\omega_2 = 24.27$ rad/s

$f_1 = 1.48$ Hz

$f_2 = 3.86$ Hz

$T_1 = 0.68$ s

$T_2 = 0.26$ s

Modes of Vibration – Manual Calculation

mode 1:

$$\begin{bmatrix} 14562.40 & -9000 \\ -9000 & 5562.40 \end{bmatrix} \begin{bmatrix} w_{2e} \\ w_{3e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{\Phi}_1 = \begin{bmatrix} 1 \\ 1.6181 \end{bmatrix}$$

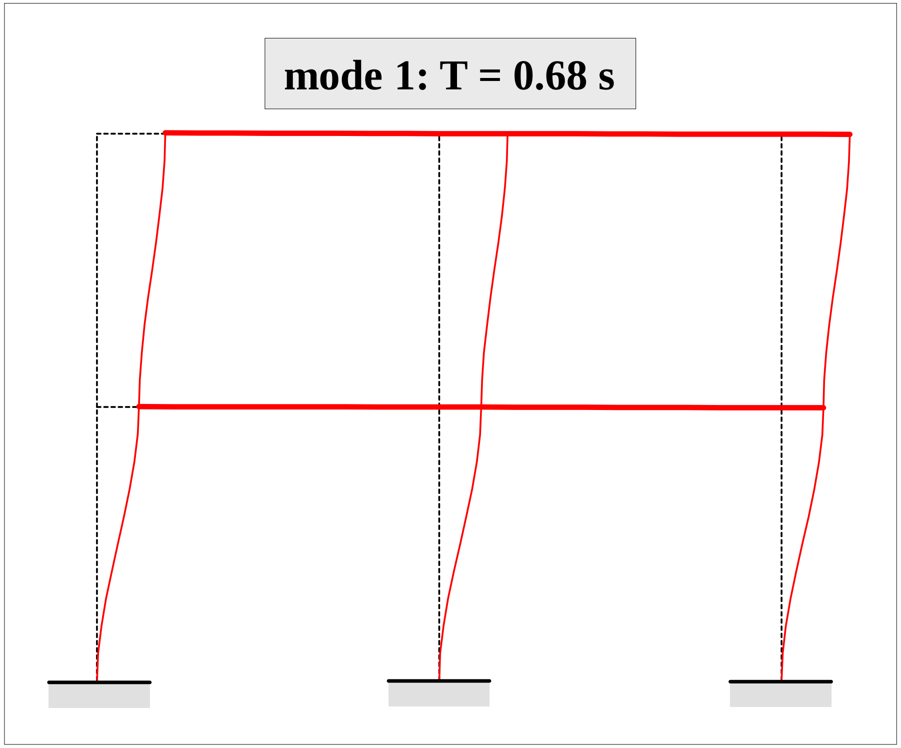
Both floors move in the same direction!

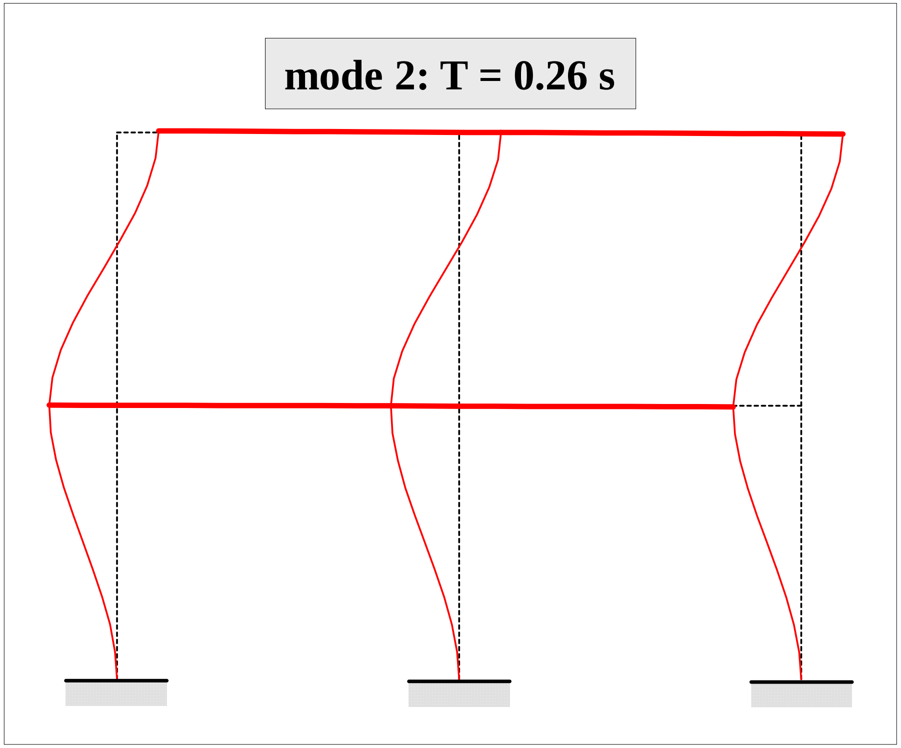
mode 2:

$$\begin{bmatrix} -5562.40 & -9000 \\ -9000 & -14562.40 \end{bmatrix} \begin{bmatrix} w_{2e} \\ w_{3e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{\Phi}_2 = \begin{bmatrix} 1 \\ -0.6180 \end{bmatrix}$$

The floors move in opposing directions!

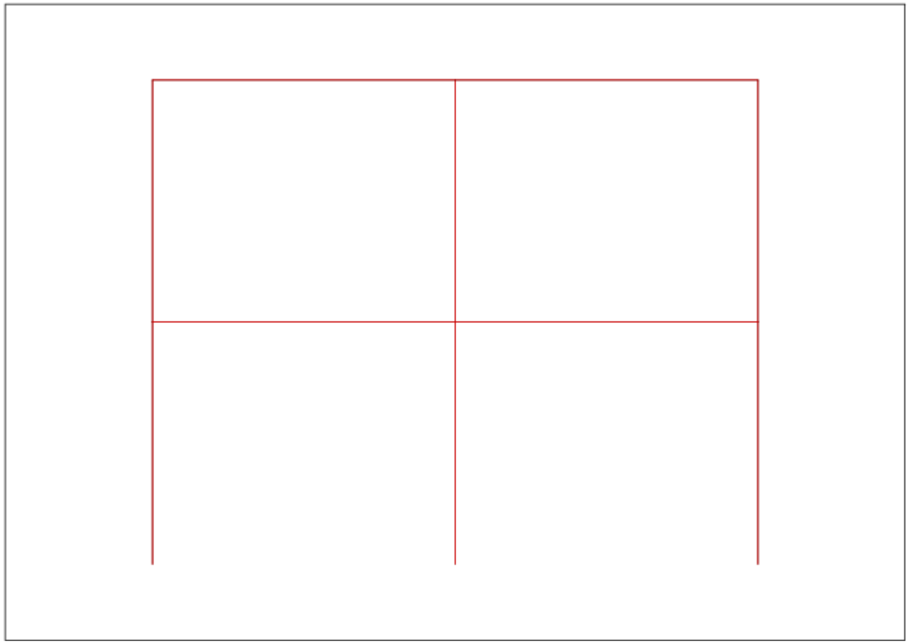
Modes of Vibration – Visual Representation



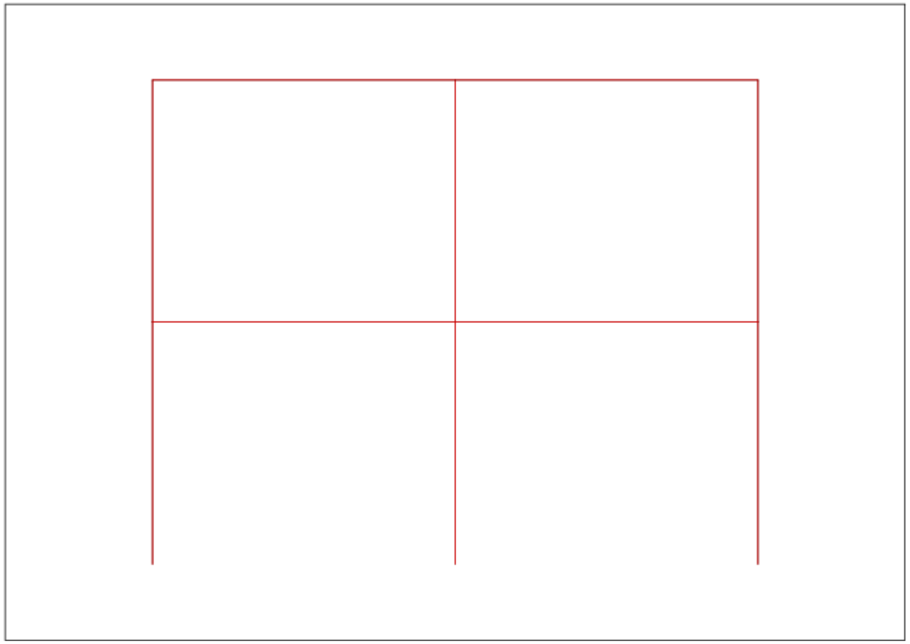


Modes of Vibration – Animation

mode 1: T = 0.68 s



mode 2: T = 0.26 s



Step 3: Transformation into Modal Space

Often in literature the eigenmodes are scaled such that the modal masses become unit masses. There is no mathematical necessity for this scaling, as we have seen in Lecture 7 "Modes Superposition Method" that the choice of the eigenvector is arbitrary. To conform to literature we perform the scaling here.

$$\tilde{m}_i = \mathbf{\Phi}_{in}^T \mathbf{M} \mathbf{\Phi}_{in} = 1$$

mode 1:

$$\mathbf{\Phi}_1^T \mathbf{M} \mathbf{\Phi}_1 = 144.73$$

$$\mathbf{\Phi}_{1n} = \frac{1}{\sqrt{144.73}} \mathbf{\Phi}_1 = \begin{bmatrix} 0.0831 \\ 0.1345 \end{bmatrix}$$

mode 2:

$$\mathbf{\Phi}_2^T \mathbf{M} \mathbf{\Phi}_2 = 55.28$$

$$\mathbf{\Phi}_{2n} = \frac{1}{\sqrt{55.28}} \mathbf{\Phi}_1 = \begin{bmatrix} 0.1345 \\ -0.0831 \end{bmatrix}$$

Modal Forces

general case

$$\tilde{\mathbf{P}}(t) = \boldsymbol{\Phi}_n^T \mathbf{P}_{quake}(t) = -\boldsymbol{\Phi}_n^T \mathbf{M}\ddot{\mathbf{V}}_{ground}(t) = \boldsymbol{\Phi}_n^T \mathbf{M}\mathbf{X}\, a_g(t)$$

our problem:

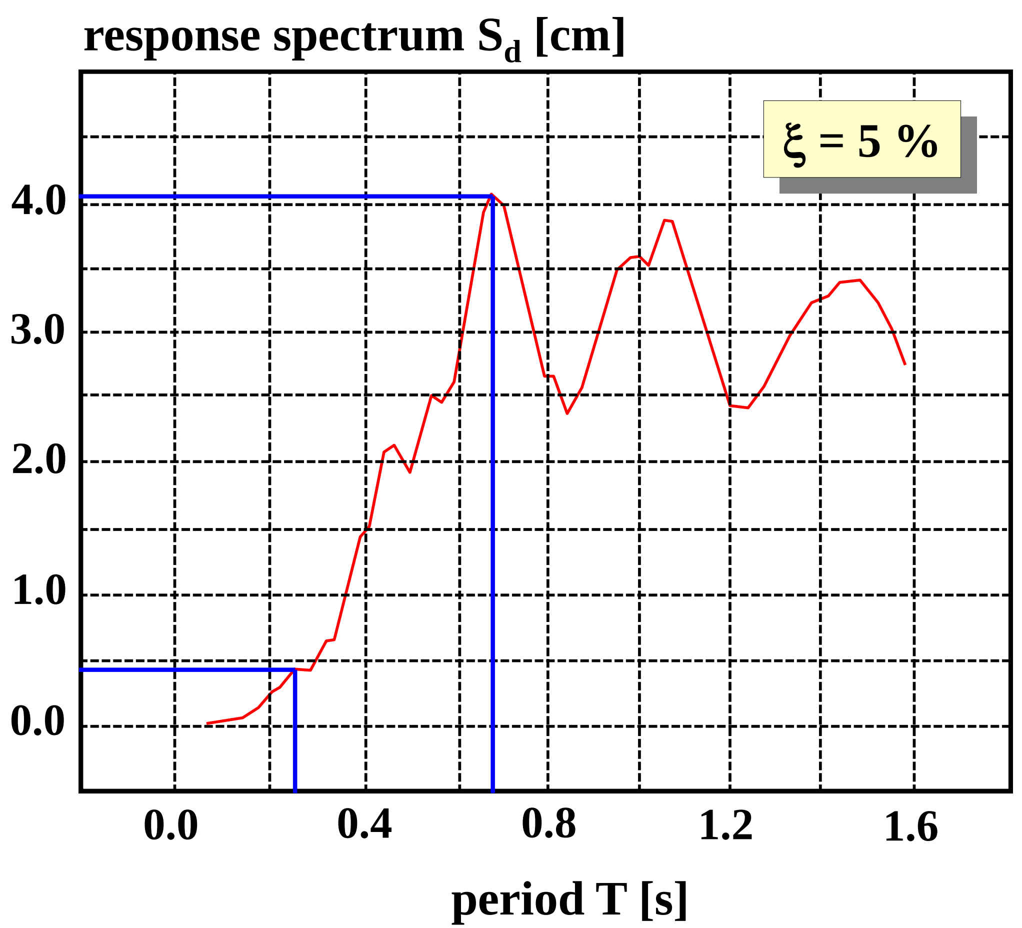
$$\ddot{\mathbf{V}}_{ground} = a_g(t)\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \tilde{\mathbf{P}} = -\begin{bmatrix}0.0831 & 0.1345\\0.1345 & -0.0831\end{bmatrix}\begin{bmatrix}40 & 0\\0 & 40\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}a_g(t)$$

$$\tilde{\mathbf{P}} = -\begin{bmatrix}8.70\\2.06\end{bmatrix}a_g(t) = -\begin{bmatrix}\tilde{\beta}_1\\\tilde{\beta}_2\end{bmatrix}a_g(t)$$

β_i tilde: modal seismic masses. The corresponding scaling factors are dimensionless. They have the same numerical values since our mode shapes led to unit modal masses.

Step 4a: Modal Response – Displacement

We obtain the modal response simply by looking up the modal values from the appropriate response spectrum.



For the modal *displacements* we use the *displacement spectrum S_d*. The spectrum on the left has been calculated from some given acceleration record and has no practical relevance here.

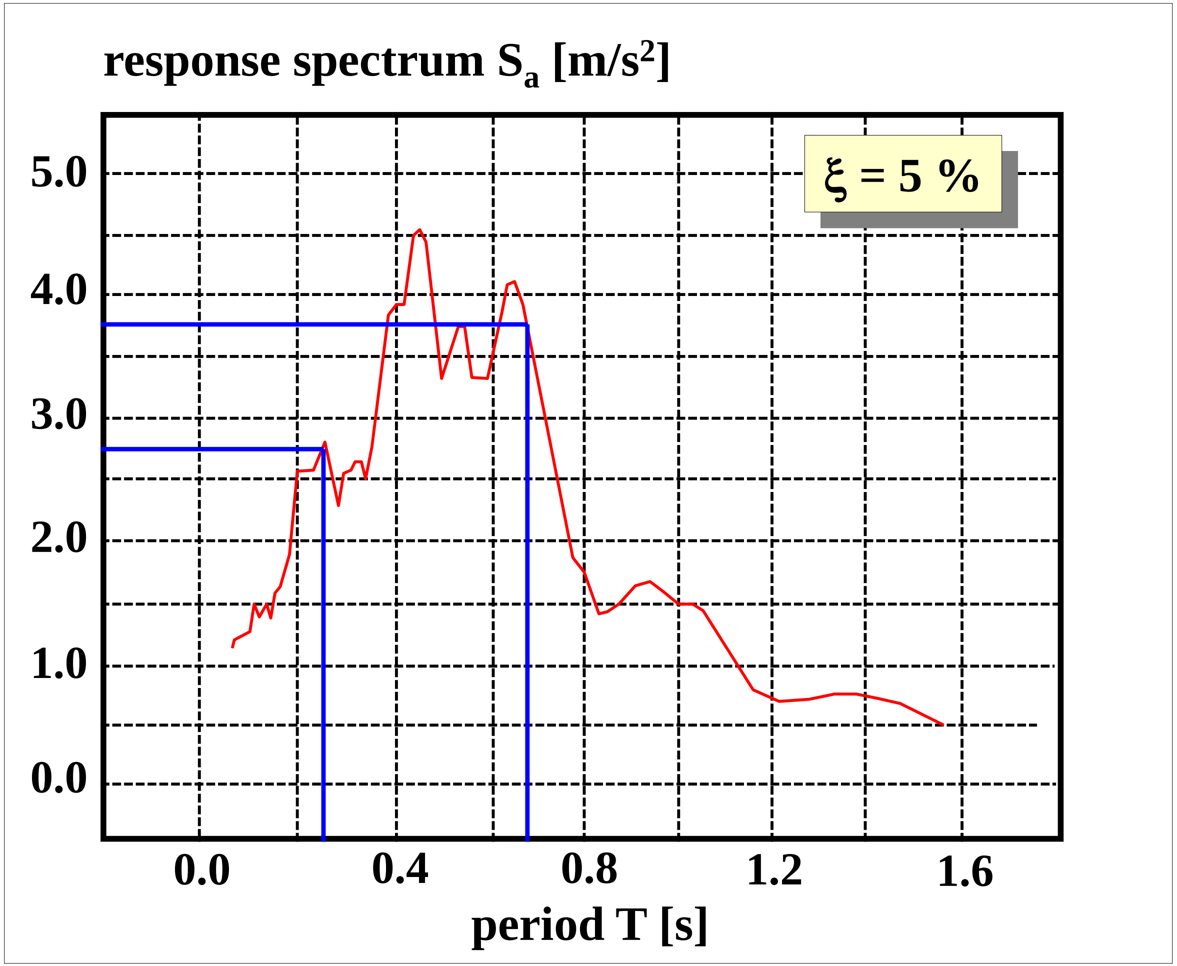
mode 1:

$T_1 = 0.68 \text{ s} \Rightarrow S_{d1} = 4.1 \text{ cm}$

mode 2:

$T_2 = 0.26 \text{ s} \Rightarrow S_{d2} = 0.44 \text{ cm}$

Step 4b: Modal Response – Acceleration



For the modal *accelerations* we use the *acceleration spectrum* S_a. The spectrum on the left is consistent with the displacement spectrum S_d since it has been calculated from the same acceleration record as S_d. It has again no practical relevance here.

mode 1:

$T_1 = 0.68$ s $\Rightarrow S_{a1} = 3.79$ m/s^2

mode 2:

$T_2 = 0.26$ s $\Rightarrow S_{a2} = 2.80$ m/s^2

Step 5: Displacements of the True Structure

maximum nodal dofs for the i-th mode:

$$\mathbf{V}_{i,\max} = \mathbf{\Phi}_{in}\,\beta_i\,S_{di}$$

mode 1:

$$\mathbf{V}_{1\max} = \begin{bmatrix} 0.0831 \\ 0.1345 \end{bmatrix} 8.70 \cdot 4.1 = \begin{bmatrix} 2.96 \\ 4.80 \end{bmatrix} \text{cm}$$

mode 2:

$$\mathbf{V}_{2\max} = \begin{bmatrix} 0.1345 \\ -0.0831 \end{bmatrix} 2.06 \cdot 0.44 = \begin{bmatrix} 0.12 \\ -0.08 \end{bmatrix} \text{cm}$$

Maximum Horizontal Displacements

Question: **How large is the maximum displacement of the true structure, i.e. the combination of modes 1 and 2?** Since the modal values do not occur at the same time, we cannot simply add up the two modal contributions. Instead we use the SRSS method which is justified since the two eigenfrequencies lie far apart. We get two ***absolute maximum values*** which means:

- Both $+w_{max}$ and $-w_{max}$ occur; we must design for both cases.
- The extreme values w_{2max} and w_{3max} do not come from a common deformation pattern. It would be incorrect to plot a deformed configuration based on these two values.

upper floor:

$$w_{3max} = \sqrt{4.80^2 + 0.08^2} = 4.80\,\mathrm{cm}$$

lower floor:

$$w_{2max} = \sqrt{2.96^2 + 0.12^2} = 2.96\,\mathrm{cm}$$

In this case the contribution of the second mode is vanishingly small!

Step 6a: Stress Resultants via Equivalent Static Loads

There are two options to calculate the inner forces. Option A takes the ***maxim accelerations*** and calculates the resulting ***inertial forces***. These inertial forces can be applied as ***static loads*** to the structure since the accelerations already contain the dynamic effects.

maximum nodal acceleration dofs for the i-th mode:

$$\ddot{\mathbf{V}}_{i,max} = \mathbf{\Phi}_{in}\, \beta_i\, S_{ai}$$

corresponding equivalent inertial mass forces:

$$\mathbf{F}_{mi,max} = \mathbf{M}\, \ddot{\mathbf{V}}_{i,max}$$

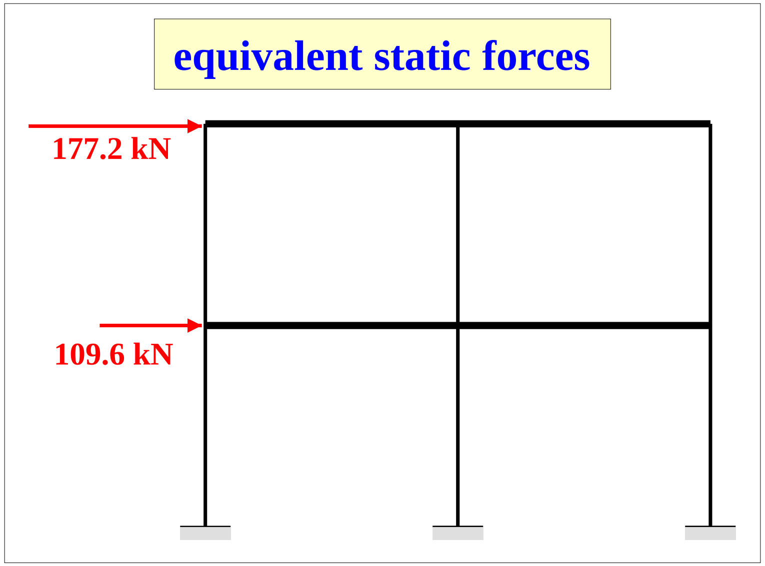
Inertial Forces Mode 1

acceleration:

$$\ddot{\mathbf{V}}_{1,\max} = \begin{bmatrix} 0.0831 \\ 0.1345 \end{bmatrix} 8.70 \cdot 3.70 = \begin{bmatrix} 2.74 \\ 4.43 \end{bmatrix} \frac{\mathrm{m}}{\mathrm{s}^2}$$

inertial force:

$$\mathbf{F}_{m1,\max} = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} \begin{bmatrix} 2.74 \\ 4.43 \end{bmatrix} = \begin{bmatrix} 109.6 \\ 177.2 \end{bmatrix} \mathrm{kN}$$



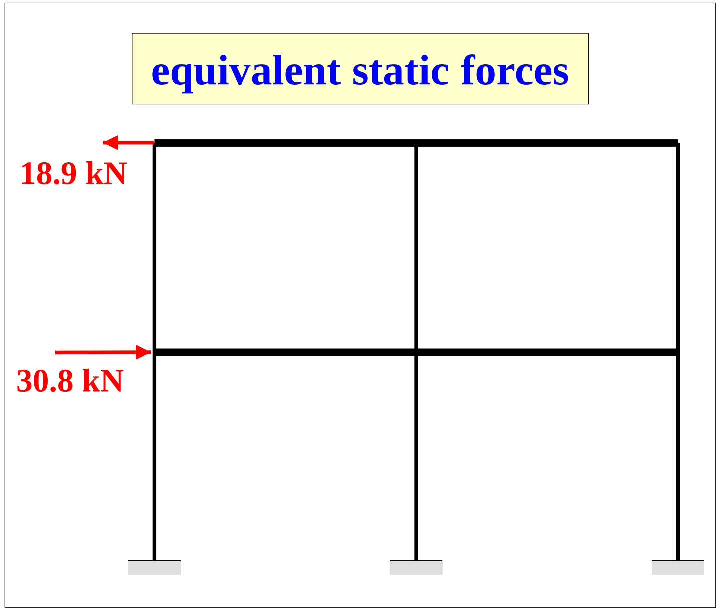
Inertial Forces Mode 2

acceleration:

$$\ddot{\mathbf{V}}_{2\max} = \begin{bmatrix} 0.1345 \\ -0.0831 \end{bmatrix} 2.06 \cdot 2.77 = \begin{bmatrix} 0.77 \\ -0.47 \end{bmatrix} \frac{\mathrm{m}}{\mathrm{s}^2}$$

inertial force:

$$\mathbf{F}_{\mathrm{m2,max}} = \begin{bmatrix} 40 & 0 \\ 0 & 40 \end{bmatrix} \begin{bmatrix} 0.77 \\ -0.47 \end{bmatrix} = \begin{bmatrix} 30.8 \\ -18.8 \end{bmatrix} \mathrm{kN}$$



Calculation of Stress Resultants

The stress resultants N, Q, M can now be computed by a *simple static analysis*. The two modes are treated as two load cases, i.e. we compute the stress resultants for *each mode separately*. To find the *total maximum inner forces* we perform an *SRSS superposition* of the mode-wise computed modal stress resultants. Again we must design to both $\pm S_{max}$.

The SRSS superposition must always be the last step. It would be wrong to compute the stresses τ or σ from the maximum inner force N_{max}, Q_{max} or M_{max}. Instead we compute the stresses mode-wise and perform the SRSS superposition directly for the modal stresses.

Stress resultants N, Q, M:
- **accelerations mode-wise**
- **loads mode-wise**
- **N, Q, M mode-wise**
- **SRSS for N_{max}, Q_{max}, M_{max}**

Stresses:
- **accelerations mode-wise**
- **loads mode-wise**
- **N, Q, M mode-wise**
- **σ, τ mode-wise**
- **SRSS for σ_{max}, τ_{max}**

Step 6b: Stress Resultants via Nodal Displacements

Option B for the calculation of inner forces takes the *maxim displacements* and computes the resulting inner forces directly from the nodal displacements of the finite elements. Option B is the natural way for an FE-program. A design response spectrum in an engineering code usually is given only as an acceleration spectrum S_a. We need, however, a displacement response spectrum S_d for option B. In absence of a separate definition of S_d we derive S_d as a *pseudo-spectrum* from S_a via the relationship:

$$\boxed{S_d = \frac{S_a}{\omega^2}}$$

Inner Force for Element 1

general relationship for the 2D beam (element i):

$$\mathbf{s}_i = \mathbf{k}_i\,\mathbf{v}_i$$

element 1 (only w_2):

mode 1 mode 2

$$\begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} -3000 \\ 6000 \\ 3000 \\ 6000 \end{bmatrix} \left[2.96 \cdot 10^{-2} \;\middle|\; 0.12 \cdot 10^{-2} \right] = \left[\begin{array}{r|r} -88.8 & -3.6 \\ 177.6 & 7.2 \\ 88.8 & 3.6 \\ 177.6 & 7.2 \end{array} \right]$$

mode 1 mode 2

SRSS:

$$M_{max} = \sqrt{177.6^2 + 7.2^2} = 177.7\,\text{kNm}$$

Seismic Design Philosophy

A building subjected to forces must equilibrate these external excitations with inner forces residing in its members. Therefore it is often a suitable means to strengthen members to reduce stresses for given stress resultants. That is not true for structures under imposed displacements. A statically indeterminate structure develops inner forces as a result of movements of the supports, while a statically determinate one, although being less stiff, deforms freely without any inner forces.

So rigid buildings cannot evade the imposed displacement and develop large inner forces which lead to damage and collapse. An earthquake design therefore aims at creating a compromise between stiffness and ductile behaviour. Often it is better to design a structure such that it can evade the imposed movements or dissipate much energy in a controlled fashion. This aim can be achieved by introducing or designing special seismic members or elements. Design options are for instance:

- **Decoupling of substructures by non-monolithic design.**
- **Introduction of specific members which dissipate energy by inelastic deformation.**
- **Active and passive damping systems. This topic will be addresses in Lecture 13 "Tuned Mass Dampers".**

A more detailed discussion of seismic design philosophies lies outside the scope of this lecture series.