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Lecture Series:

Structural Dynamics

Lecture 10:

Finite Element Formulation

Overview

- Principle of virtual work
- Discretization
- Examples:
  - truss element
  - beam element
- Consistent and lumped mass models
- Rayleigh Damping

Principle of Virtual Work

Static problem:

Total work = work of (external loads + internal stresses)

$$\delta W = \int_V \delta \mathbf{u}^T \mathbf{p} \, dV - \int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dV = 0$$

Dynamic problem:

Total work = work of (external loads + inertial mass forces + internal stresses)

$$\delta W = \int_V \delta \mathbf{u}^T \mathbf{p} \, dV - \int_V \delta \mathbf{u}^T \mathbf{f}_m \, dV - \int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dV = 0$$

Review: Discretization of the Static Problem

Displacement interpolation:

$$\mathbf{u} = \boldsymbol{\Omega}\,\mathbf{v}$$

Ω: shape functions, v: nodal dofs

Strain interpolation:

$$\boldsymbol{\varepsilon} = \mathbf{D}_k\mathbf{u} = \mathbf{D}_k\boldsymbol{\Omega}\,\mathbf{v} = \mathbf{B}\,\mathbf{v}$$

D_k: kinematic operator

Stress interpolation:

$$\boldsymbol{\sigma} = \mathbf{E}\,\boldsymbol{\varepsilon} = \mathbf{E}\mathbf{B}\,\mathbf{v}$$

E: elasticity matrix

Discretization of the Work Principle

Weak form of the equilibrium condition:

$$\delta W = \delta \mathbf{v}^T \left\{ \int_V \boldsymbol{\Omega}^T \mathbf{p} \, dV - \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV \mathbf{v} \right\} = 0$$

Element stiffness equation:

$$\delta \mathbf{v}^T \left[\mathbf{q} - \mathbf{k} \, \mathbf{v} \right] = 0 \quad \Rightarrow \quad \mathbf{k} \, \mathbf{v} = \mathbf{q}$$

Element load vector:

$$\mathbf{q} = \int_V \boldsymbol{\Omega}^T \mathbf{p} \, dV$$

Element stiffness matrix:

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \, dV$$

Treatment of Inertial Forces

Work of inertial forces:

$$\delta W_m = \int_V \delta \mathbf{u}^T \mathbf{f}_m \, dV$$

NEWTON's law

$$\mathbf{f}_m = \rho \, \ddot{\mathbf{u}}$$

Discretization:

$$\mathbf{u} = \boldsymbol{\Omega} \, \mathbf{v}$$

$$\mathbf{f}_m = \rho \, \boldsymbol{\Omega} \, \ddot{\mathbf{v}}$$

Discretized work term:

$$\delta W_m = \delta \mathbf{v}^T \int_V \boldsymbol{\Omega}^T \rho \, \boldsymbol{\Omega} \, dV \, \ddot{\mathbf{v}} = \delta \mathbf{v}^T \mathbf{m} \, \ddot{\mathbf{v}}$$

Element mass matrix:

$$\mathbf{m} = \int_V \boldsymbol{\Omega}^T \rho \, \boldsymbol{\Omega} \, dV$$

Element Equation of Motion

Principle of virtual work:

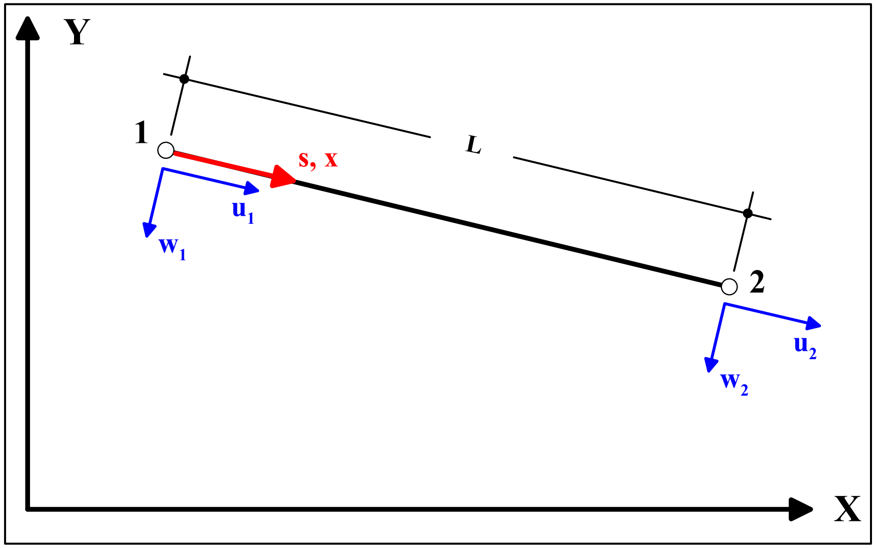
$$\delta \mathbf{v}^{T}\left[\mathbf{q}-\mathbf{m}\,\ddot{\mathbf{v}}-\mathbf{k}\,\mathbf{v}\right] = 0$$

Euler-Lagrange equation

Element equation of motion:

$$\mathbf{m}\,\ddot{\mathbf{v}}+\mathbf{k}\,\mathbf{v} = \mathbf{q}(t)$$

Example 1: Plane Truss Element



geometry:

$$dx = L\,ds$$

nodal dofs:

$$\mathbf{v}_u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \mathbf{v}_w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

shape functions:

$$u(s) = u_1(1-s) + u_2 s = \begin{bmatrix} 1-s & s \end{bmatrix} \mathbf{v}_u$$

$$w(s) = w_1(1-s) + w_2 s = \begin{bmatrix} 1-s & s \end{bmatrix} \mathbf{v}_w$$

We can use the same linear shape functions which we have already used for the derivation of the stiffness matrix.

Derivation of the Mass Matrix for the u-Direction

General expression for a truss:

$$\mathbf{m}_u = \int_V \boldsymbol{\Omega}_u^T \rho \boldsymbol{\Omega}_u \, dV = \int_L \int_A \rho \boldsymbol{\Omega}_u^T \boldsymbol{\Omega}_u \, dA \, dx = \rho A \int_L \boldsymbol{\Omega}_u^T \boldsymbol{\Omega}_u \, dx$$

μ: mass per unit length of the truss

$$\boxed{\mu = \rho A}$$

For the linear shape function:

$$\int_V \boldsymbol{\Omega}_u^T \boldsymbol{\Omega}_u \, dx = \int_0^1 \begin{bmatrix} 1-s \\ s \end{bmatrix} \begin{bmatrix} 1-s & s \end{bmatrix} L \, ds = \frac{L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Element Mass Matrix

For the 2 directions:

$$\mathbf{m}_u = \frac{M}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow \mathbf{m}_w = \frac{M}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assemble partial matrices to local element mass matrix:

$$\mathbf{m} = \frac{M}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

mass of the element:

$$M = \rho L A$$

Transformation to Global DOFs

Transformation of displacements:

$$\mathbf{v}_{\text{local}} = \mathbf{T}\,\mathbf{v}_{\text{global}}$$

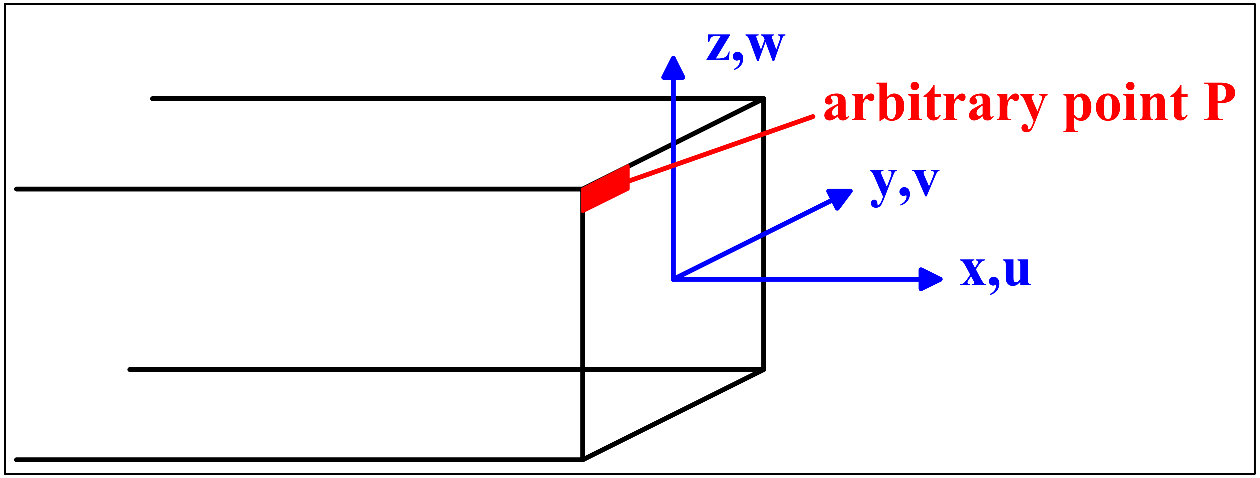
We have derived a transformation rule for the element stiffness matrix. This general rule holds for all types of matrices, also for the mass matrix:

$$\mathbf{m}_{\text{global}} = \mathbf{T}^{\text{T}}\,\mathbf{m}_{\text{local}}\,\mathbf{T}$$

A body has the same translational inertia in all directions, while the rotational inertia (the mass moments of inertia Θ) depends on the geometrical shape and is different in different directions. The truss element is a special case because we have no rotational degrees of freedom. The mass matrix is therefore invariant with respect to rotations of the coordinate system:

$$\mathbf{m}_{\text{global}} = \mathbf{m}_{\text{local}}$$

Example 2: Beam Element



Kinematics for the beam element without distortion:

$$u^*(x,y,z) = u(x) + \varphi_y(x)\cdot z - \varphi_z(x)\cdot y$$

$$v^*(x,y,z) = v(x) - \varphi_x(x)\cdot z$$

$$w^*(x,y,z) = w(x) + \varphi_x(x)\cdot y$$

The cross section remains plane. We can express the displacements of an arbitrary point within the cross section by the deformation variables of the beam axis: extensional displacement u, bending deflections v and w, torsional rotation φ_x and bending rotation φ_y and φ_z.

Virtual Work of the Inertial Forces

Virtual work for the beam continuum:

$$\delta W_m = \int_V (\delta u^{*T} \rho \, \ddot{u}^* + \delta v^{*T} \rho \, \ddot{v}^* + \delta w^{*T} \rho \, \ddot{w}^*) \, dV$$

$$\delta W_m = \delta W_{mu^*} + \delta W_{mv^*} + \delta W_{mw^*}$$

Introduce kinematics of the beam:

$$\delta W_{mu^*} = \rho \int_V \left[(\delta u + \delta\varphi_y z - \delta\varphi_z y)^T (\ddot{u} + \ddot{\varphi}_y z - \ddot{\varphi}_z y) \right] dA \, dx$$

$$\delta W_{mv^*} = \rho \int_V \left[(\delta v - \delta\varphi_x z)^T (\ddot{v} - \ddot{\varphi}_x z) \right] dA \, dx$$

$$\delta W_{mw^*} = \rho \int_V \left[(\delta w + \delta\varphi_x y)^T (\ddot{w} + \ddot{\varphi}_x y) \right] dA \, dx$$

Cross-Sectional Moments

Assumption 1: **reference system lies in the center of gravity**
static moments are zero

$$S_y = \int_A z\,dA = 0 \qquad S_z = \int_A y\,dA = 0$$

Assumption 2: **reference system is oriented along principal axes**
deviational moment of inertia is zero

$$I_{yz} = -\int_A yz\,dA = 0$$

Remaining cross-sectional moments

$$A = \int_A dA \qquad I_{zz} = \int_A y^2\,dA \qquad I_{yy} = \int_A z^2\,dA \qquad I_p = I_{yy} + I_{zz}$$

Specialised Virtual Work

Introduce cross-sectional moments:

$$\delta W_m = \rho \int_L (\delta u\, A\, \ddot{u} + \delta v\, A\, \ddot{v} + \delta w\, A\, \ddot{w} + \delta\varphi_x\, I_p\, \ddot{\varphi}_x + \delta\varphi_y\, I_{yy}\, \ddot{\varphi}_y + \delta\varphi_z\, I_{zz}\, \ddot{\varphi}_z)\, dx$$

BERNOULLI hypothesis: no shear deformations

$$\varphi_y = -w'$$

$$\varphi_z = +v'$$

Interpolation functions must be chosen for:
- **the longitudinal displacement u,**
- **the torsional rotation φ_x,**
- **the bending displacements v and w.**

Part 1+2: Extensional and Torsional Vibration

The extensional deformation can be treated exactly as in the truss element. The work for φ_x is formally identical to the work for u. So we can choose the same linear shape function to interpolate between the two nodal degrees of freedom φ_{x1} and φ_{x2}. We can copy the matrix for u and substitute I_p for A:

$$\mathbf{m}_u = \frac{M}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{m}_{\varphi_x} = \frac{\rho I_p L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

This matrix for u is only valid for the degrees of freedom u_1 and u_2, but not for v_1, v_2, w_1, w_2, since the shape functions for v and w are cubic!

Part 3-1: Translational Bending Vibration

The shape functions for bending in the xz-plane (degrees of freedom w_1, w_2, φ_{y1}, φ_{y2}) can be copied from the derivation of the linear stiffness matrix:

$$\Omega = \begin{bmatrix} 1 - 3s^2 + 2s^3 \\ (-s + 2s^2 - s^3)L \\ 3s^2 - 2s^3 \\ (s^2 - s^3)L \end{bmatrix}$$

$$\delta W_m = \rho \int_L \delta w \, A \, \ddot{w} \, dx$$

$$\mathbf{m}_{w1} = \frac{M}{420} \begin{bmatrix} 156 & -22L & 54 & 13L \\ -22L & 4L^2 & -13L & -3L^2 \\ 54 & -13L & 156 & 22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}$$

Part 3-2: Rotational Bending Vibration

$$\delta W_m = \int_L \delta\varphi_y \, I_{yy} \, \ddot{\varphi}_y \, dx$$

$$\mathbf{m}_{w2} = \frac{\rho I_{yy}}{30L} \begin{bmatrix} 36 & -3L & -36 & -3L \\ -3L & 4L^2 & 3L & -L^2 \\ -36 & 3L & 36 & 3L \\ -3L & -L^2 & 3L & 4L^2 \end{bmatrix}$$

The matrix for v can be derived in a wholly analogous manner! All matrices together comprise the element mass matrix of the beam.

Consistent Mass Matrices

The same interpolation function has been used for the stiffness and the mass matrix.

↓

Both matrices are based on the same assumptions.

↓

They are consistent with respect to the displacement interpolation.

↓

Consistent Mass Matrix (CMM)

Advantages/Disadvantages of CMM

Advantages:

Each nodal degree of freedom is automatically assigned its correct mass. In particular the rotational degrees of freesdom are also given inertial moments of mass.

Disadvantages:

The storage requirement is high: a system mass matrix has the same storage image (band matrix, skyline matrix, sparse matrix, ...) and takes up the same storage space.

Alternative: Lumped Mass Matrix

In a lumped mass matrix (LMM) the element masses are "lumped" into the nodes, i.e. we have pure nodal masses. The system mass matrix becomes a diagonal matrix and can be stored as a vector.

The storage requirement is greatly reduced, almost to zero with respect to a "normally" populated matrix.

Example: truss element

$$\mathbf{m}_{\text{lumped}} = \frac{M}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \mathbf{m}_{\text{lumped}} = \frac{M}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Problems with Lumped Mass Matrix

The translational masses are relatively easy to define, the rotational masses are not obvious. Often they are neglected.

The lumped mass matrix is not positive definite for $M_{rot} = 0$!

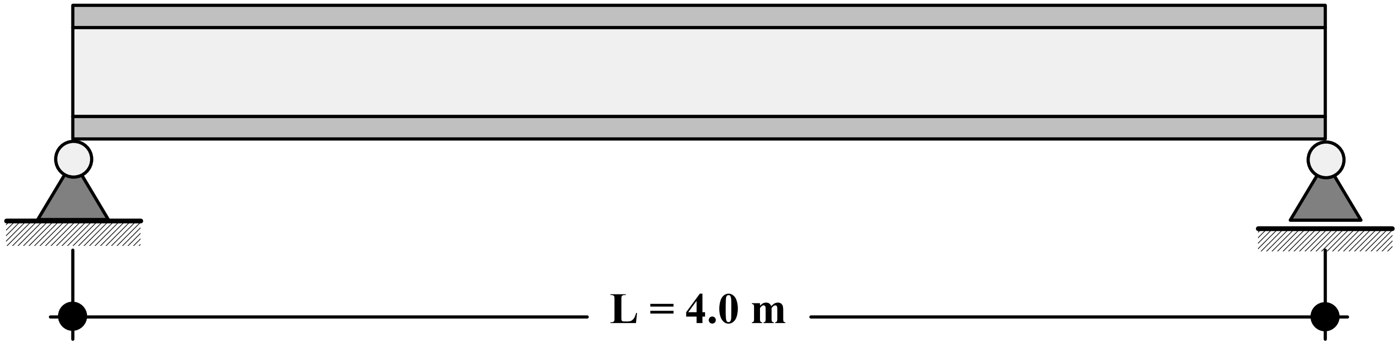
- **Eigenfrequency analysis:**

No problem, there can be zeroes on the diagonal. Only: the number of eigenfrequencies is limited to the rank of the mass matrix.

- **Direct time integration:**

Fatal defect since for the computation of the initial acceleration it is necessary to solve a system of equations with M as the coefficient matrix. Therefore it must not be singular! $\Rightarrow$ Rotational masses are mandatory!

Example I: Simply Supported Beam



HE-A 500: $A = 198\ \text{cm}^2$
$I = 86970\ \text{cm}^4$
$E = 21000\ \text{kN/cm}^2$
$m = 0.1584\ \text{to/m}$

analytical solution for the i-th eigenfrequency:

$$f_i = \frac{\pi i^2}{2L^2}\sqrt{\frac{EI}{m}}$$

$\Rightarrow$

$f_1 = 105.4185\ \text{Hz}$
$f_2 = 421.6739\ \text{Hz}$
$f_3 = 948.7662\ \text{Hz}$

Example I: Simply Supported Beam

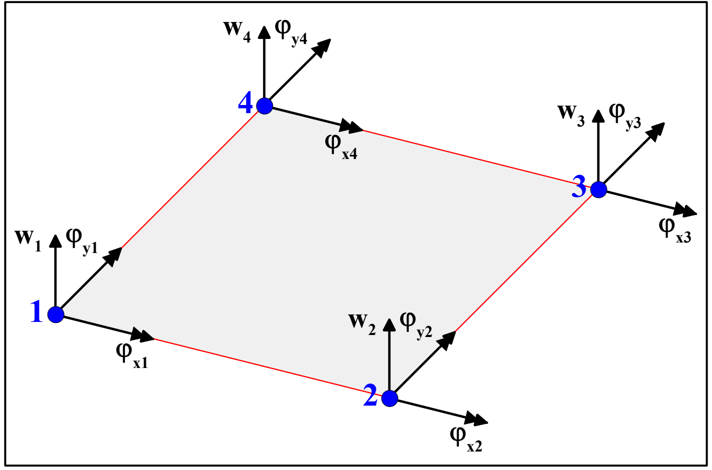
eigenfrequency 1 [Hz] - analytical solution: 105.4185 Hz						
	1	2	4	8	16	32
consistent	117.01	105.83	105.45	105.42	105.42	105.42
lumped	-	104.65	105.39	105.42	105.42	105.42

eigenfrequency 2 [Hz] - analytical solution: 421.6739 Hz						
	1	2	4	8	16	32
consistent	536.19	468.02	423.34	421.78	421.68	421.67
lumped	-	-	418.61	421.54	421.67	421.67

eigenfrequency 3 [Hz] - analytical solution: 948.7662 Hz						
	1	2	4	8	16	32
consistent	-	1176.4	966.10	949.99	948.84	948.77
lumped	-	-	888.80	947.02	948.68	948.76

FE-Models for Surface-Like Structures

generic slab element with 4 generic nodes

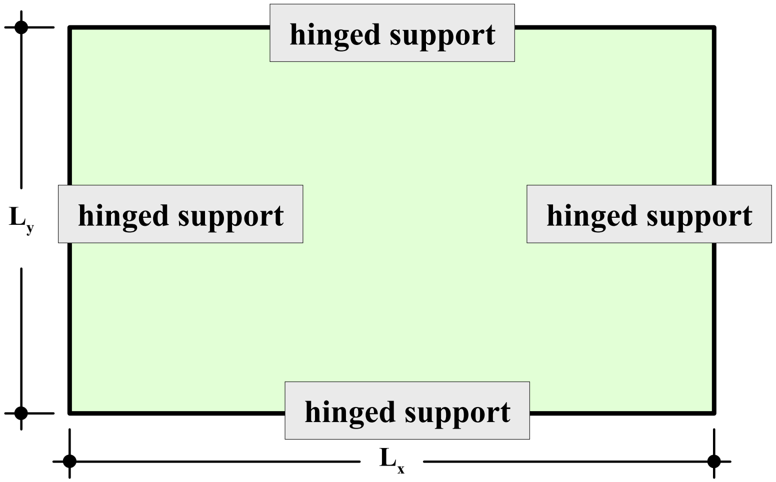


There exist a multitude of slab and shell element types, for shell theories without (KIRCHHOFF/LOVE-type theories) and with (MINDLIN/REISSNER-type theories) shear deformations. We use here the element type ASE4 developed by U. Montag (*Konzepte zur Effizienzsteigerung numerischer Simulationsalgorithmen für elastoplastische Deformationsprozesse*, Dissertation, Bochum 1997). It takes shear deformations into account via a so-called *assumed strain formulation*. The element uses bilinear shape functions for the three displacement components and the two rotations.

Since we do not know, except for these rather general mathematical properties, how the element performs, we have to run a set of suitable benchmarks for which we have either analytical solutions or other numerical reference solutions. With such benchmarks we can gain insight into the strengths or weaknesses of the element types in our element library.

The solution for a rectangular slab, supported along all four sides by hinged supports, is known for a KIRCHHOFF slab theory. The influence of the shear deformations, however, are small for a thin slab so we can use this solution as benchmark. We expect the ASE4-solution to be marginally softer then the KIRCHHOFF solution. The mass stays the same, so the eigenfrequencies without shear deformations are somewhat larger than the ones computed by a shear theory.

Example II: Hinged Rectangular Slab



dimensions:	**$L_x = 8.0$ m, $L_y = 4.0$ m,**
material properties:	**$E = 3.0\ 10^7$ kN/m^2** **$\nu = 0.2$** **$\rho = 2.5$ to/m^3**
slab thickness:	**$h = 0.1$ m**

analytical solution for a KIRCHHOFF slab theory

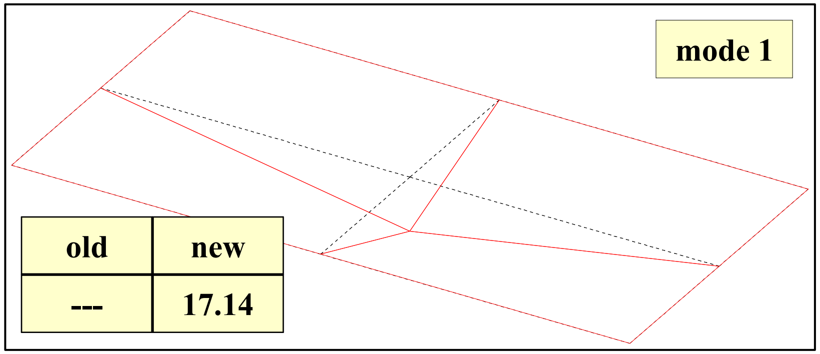
$$f = \frac{\omega}{2\pi} = \frac{\pi}{2}\left\{\left[\frac{N_x}{L_x}\right]^2 + \left[\frac{N_y}{L_y}\right]^2\right\}\sqrt{\frac{B}{\rho h}}$$

There exist an infinite number of eigenmodes with different wave numbers N_x and N_y.

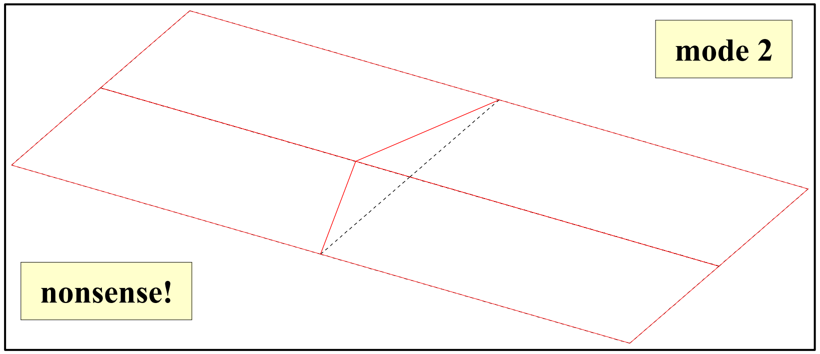
analytical solution (without shear strains)

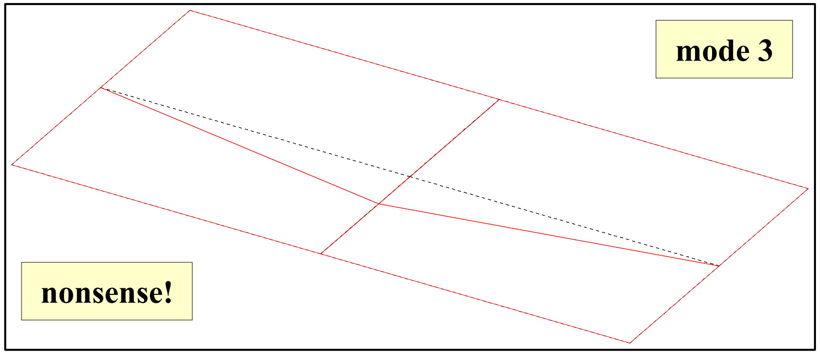
waves	eigenfrequency f [Hz]				
	12.525	20.040	32.565	42.585	50.100
N_x	1	2	3	1	2
N_y	1	1	1	2	2

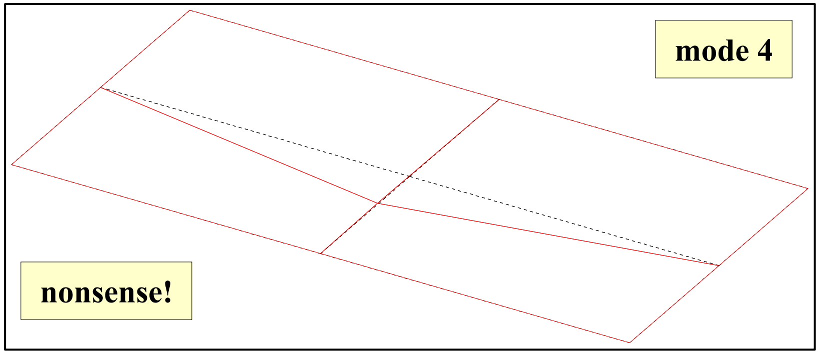
Element Mesh: 2 x 2

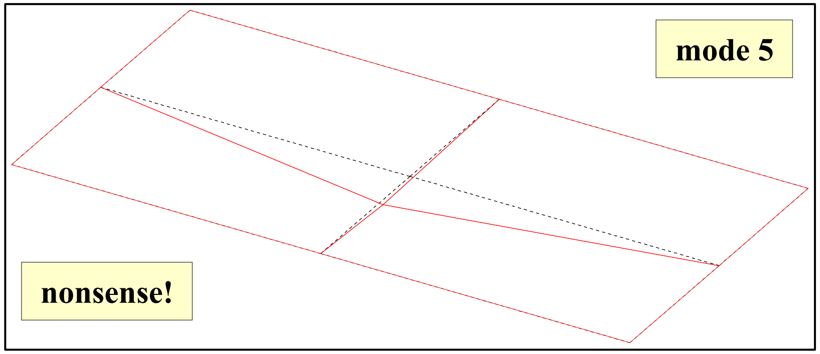


old	new
---	17.14



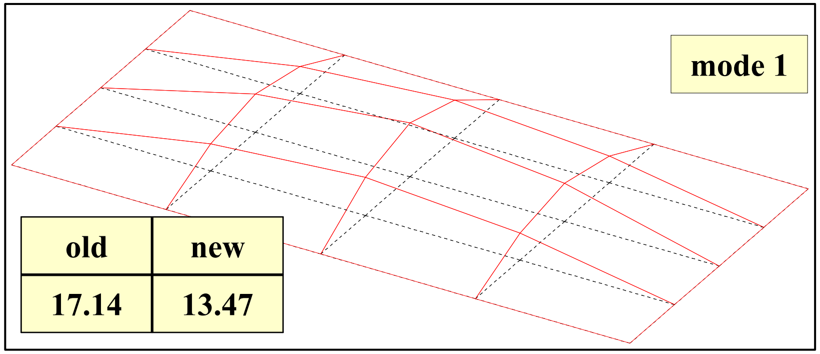




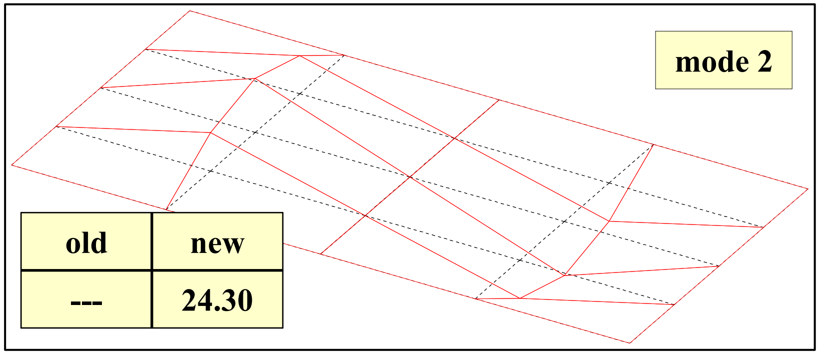


Only the 1st mode models the physical reality, i.e. a vibration mode with one wave in both directions. The other modes are nonsense modes resulting from rotational degrees of freedom. These modes have to be discarded. The coarse 2x2 mesh can only capture, albeit badly, one single mode.

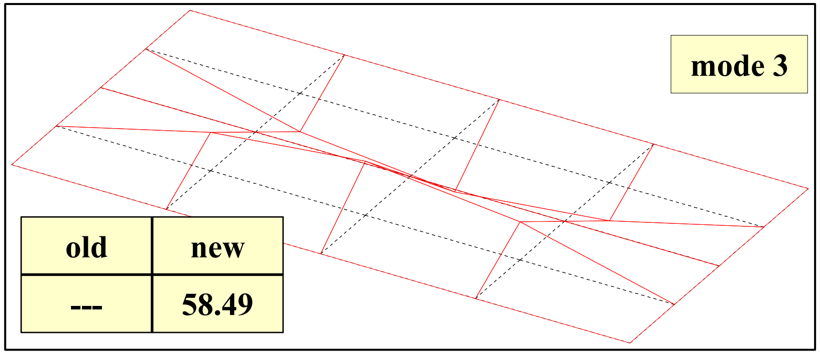
Element Mesh: 4 x 4



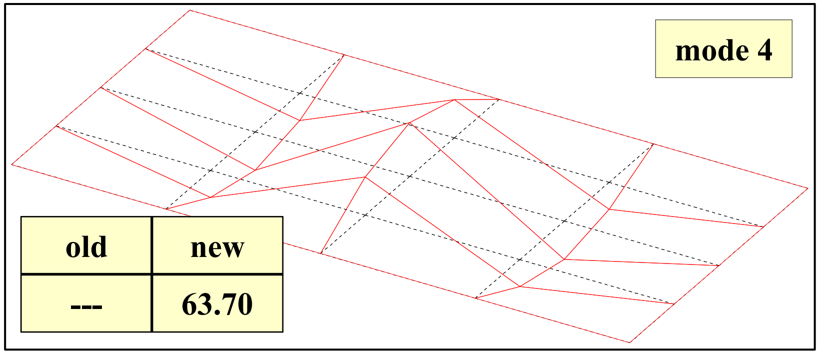
old	new
17.14	13.47



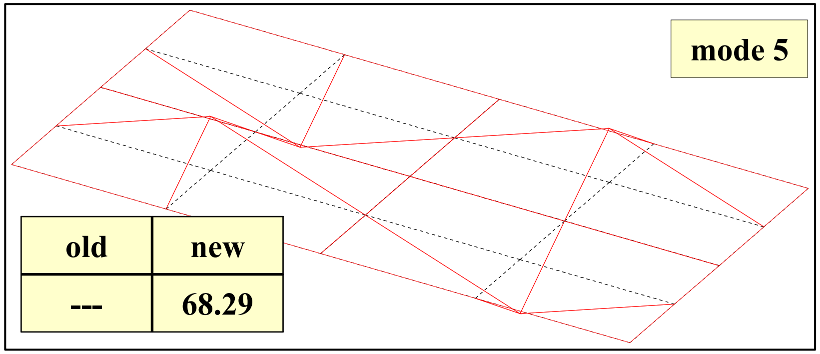
old	new
---	24.30



old	new
---	58.49



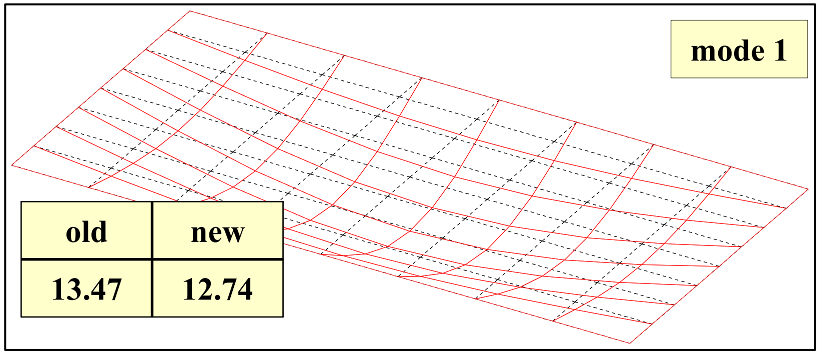
old	new
---	63.70



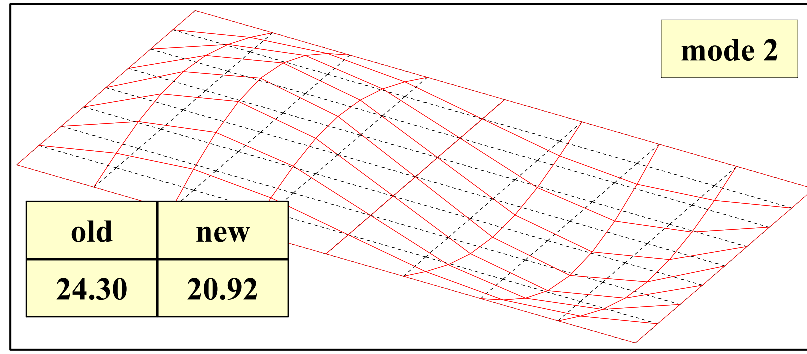
old	new
---	68.29

Now all five modes are physically significant. For mode 1 we can test the convergence: the change form mesh02 to mesh04 is significant, so f_1 from mesh02 was too inaccurate. Modes 3 and 4 are switched with respect to the analytical solution. The non-smoothness of the mode shape reveals their inaccuracy.

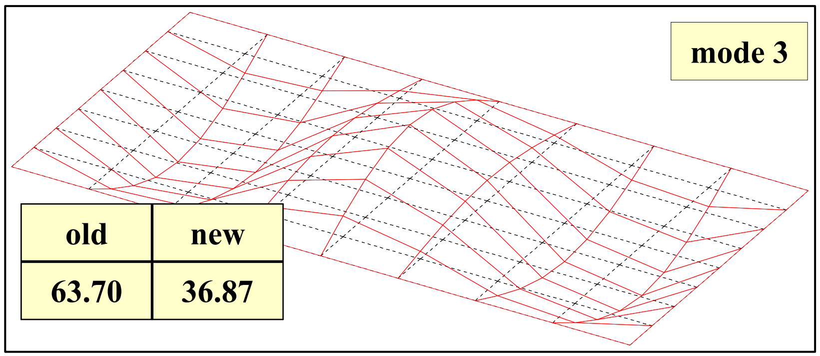
Element Mesh: 8 x 8



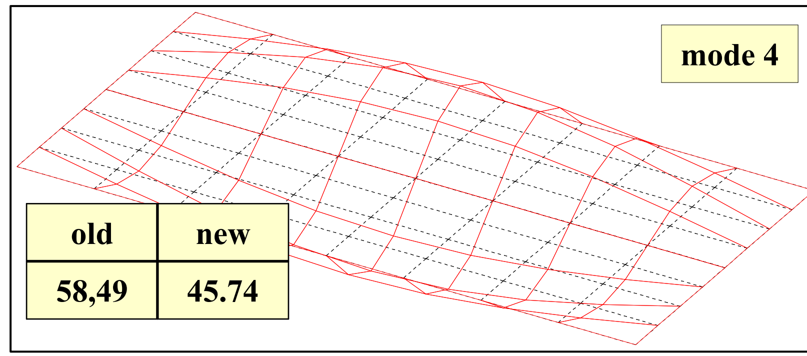
old	new
13.47	12.74



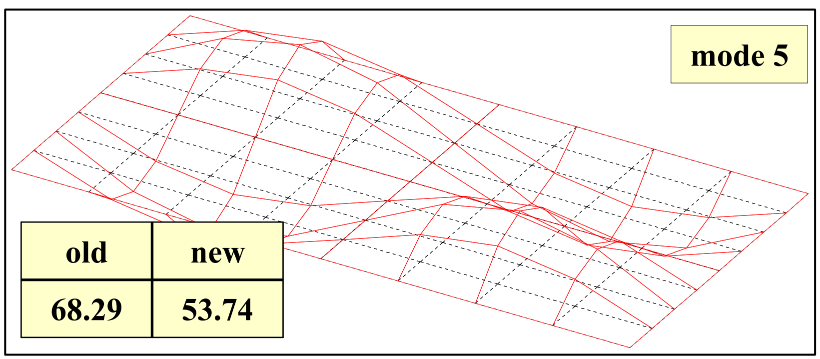
old	new
24.30	20.92



old	new
63.70	36.87



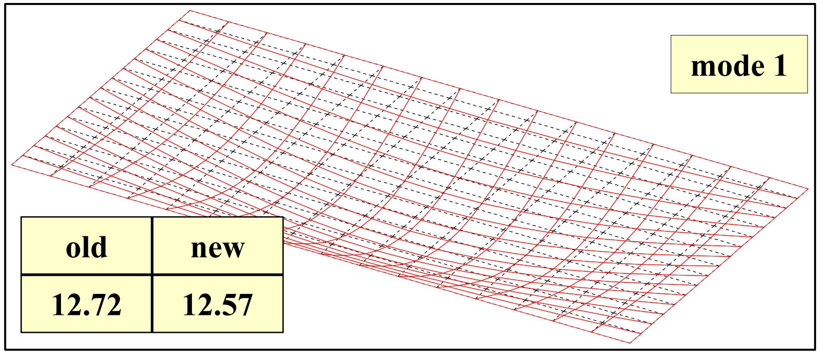
old	new
58,49	45.74



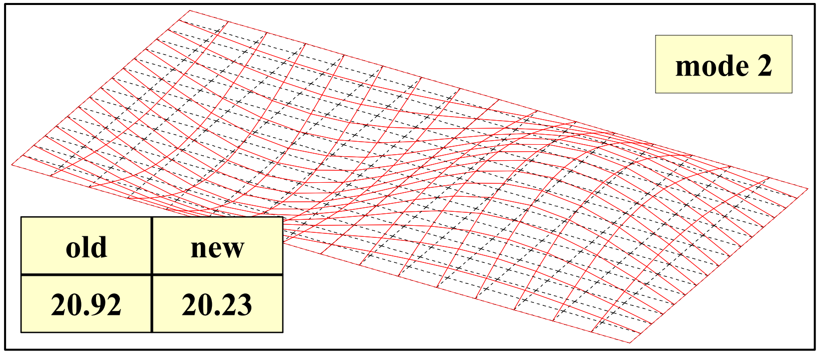
old	new
68.29	53.74

The sequence of modes now follows the correct sequence of the analytical solution. The shapes are getting smoother. For mode 1 we are getting nearer to convergence (5.7 % change from mesh04 to mesh08), while the higher modes still show larger changes.

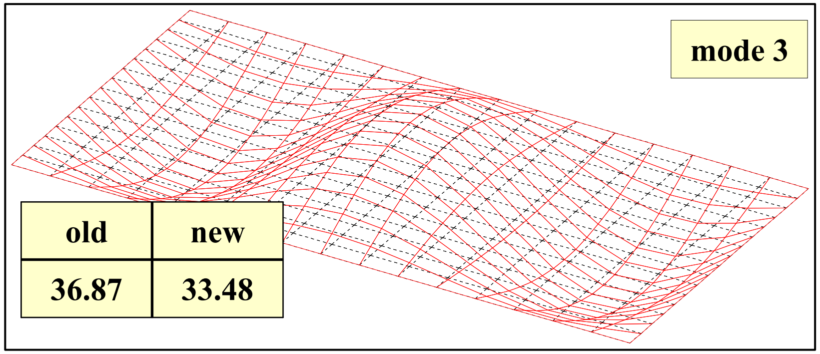
Element Mesh: 16 x 16



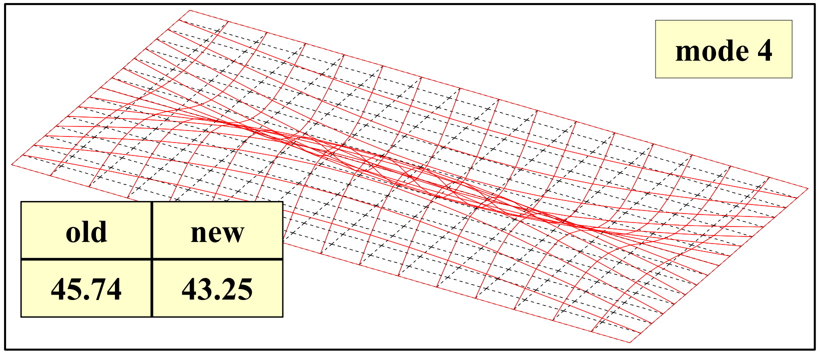
old	new
12.72	12.57



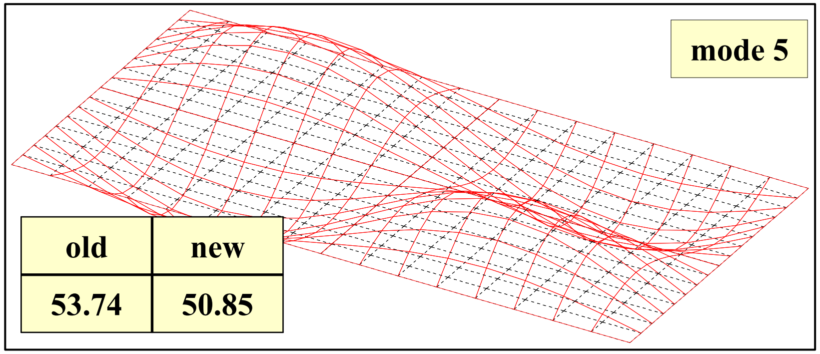
old	new
20.92	20.23



old	new
36.87	33.48



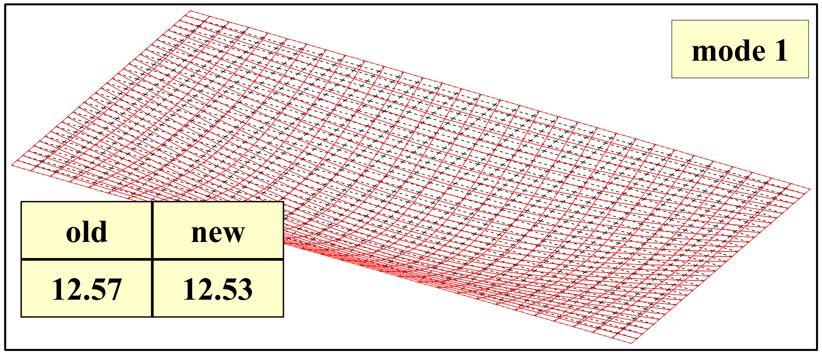
old	new
45.74	43.25



old	new
53.74	50.85

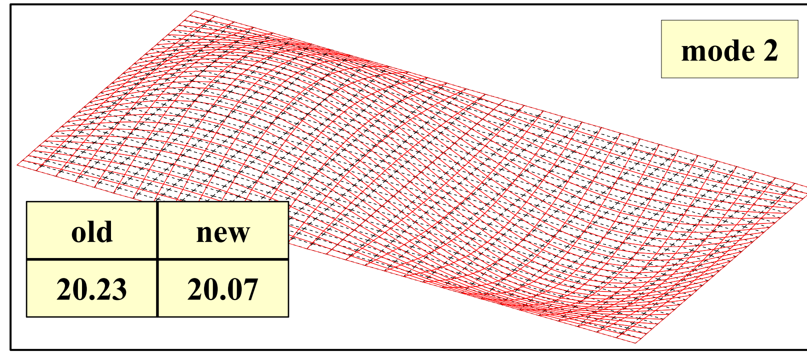
All modes converge. The higher modes converge slower than the lower ones, since it is more difficult to capture their more complex wave patterns by the bilinear shape function within each element.

Element Mesh: 32 x 32



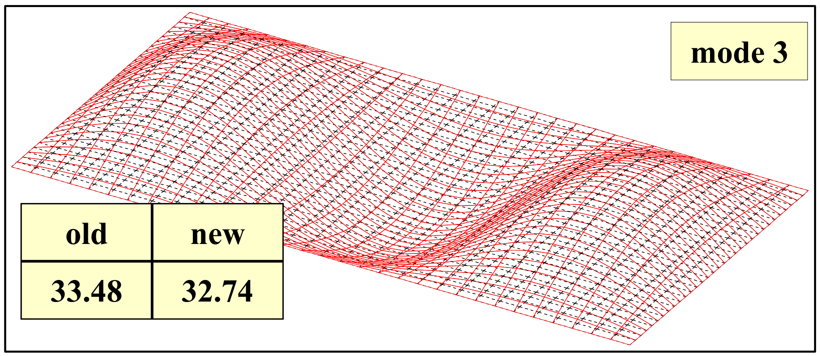
mode 1

old	new
12.57	12.53



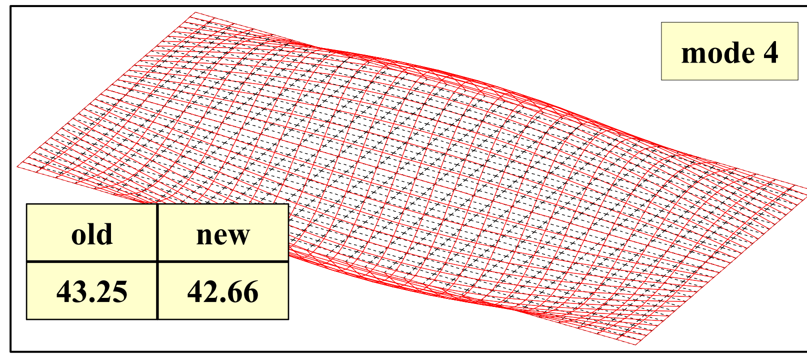
mode 2

old	new
20.23	20.07



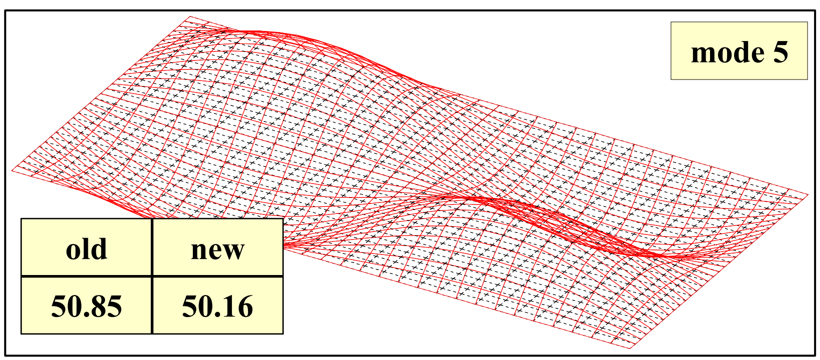
mode 3

old	new
33.48	32.74



mode 4

old	new
43.25	42.66

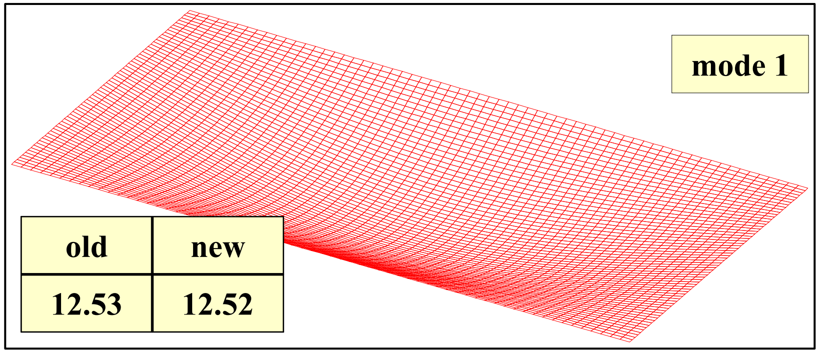


mode 5

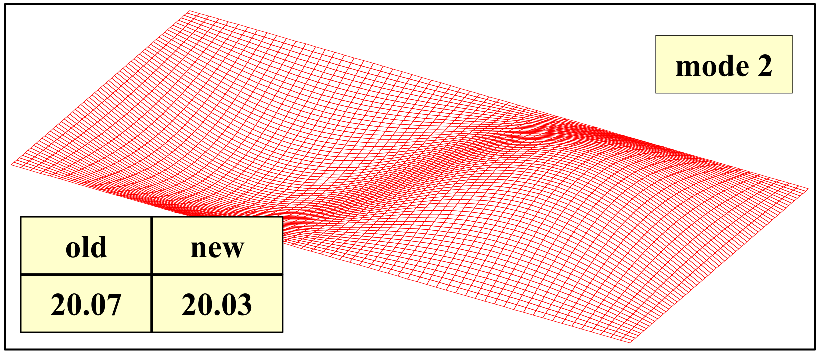
old	new
50.85	50.16

The maximum error is down to 1.4 % - we run just one mesh refinement more to attain an accuracy which lies in the range of the shear deformations, so we can check whether we converge to frequencies which lie a little bit below the ones from the KIRCHHOFF slab theory.

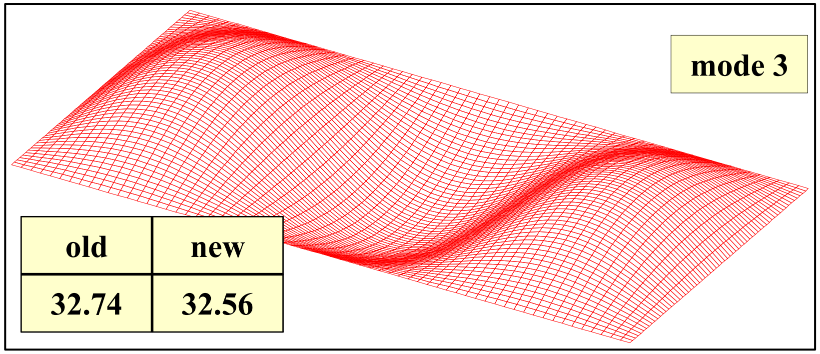
Element Mesh: 64 x 64



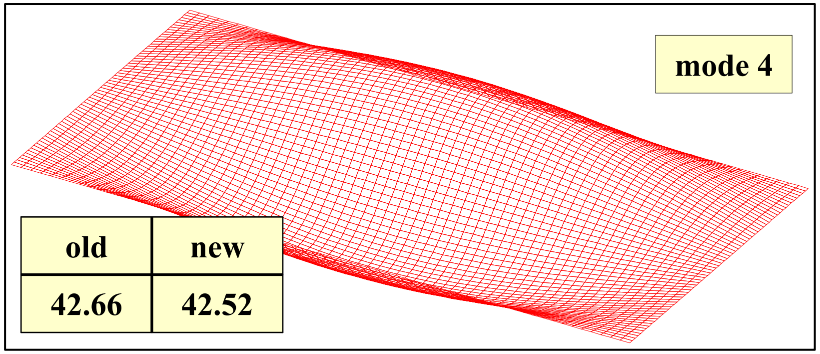
old	new
12.53	12.52



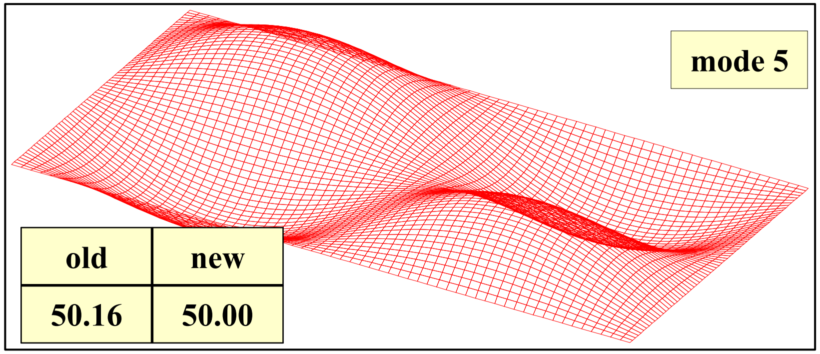
old	new
20.07	20.03



old	new
32.74	32.56



old	new
42.66	42.52



old	new
50.16	50.00

analytical solution (without shear strains)					
f [Hz]	12.525	20.040	32.565	42.585	50.100
N_x	1	2	3	1	2
N_y	1	1	1	2	2

Summary

The convergence test on the previous pages is only valid for the element type ASE4. We must repeat the test if we want to use a different element type where we lack experience regarding its performance. Below are tabled the results for the classic nonconforming 4-node slab element with cubic shape functions with KIRCHHOFF slab theory. The nonconformity introduces additional relative rotations along the element borders which reduce the element stiffness. The approximation is therefore too soft and we converge, except for the 1st super-coarse mesh, from below to the analytical solution.

convergence test of the nonconforming KIRCHHOFF element					
mesh	f_1 [Hz]	f_2 [Hz]	f_3 [Hz]	f_4 [Hz]	f_5 [Hz]
02	11.070	17.933	35.715	46.666	58.553
04	12.089	18.868	31.120	40.788	44.279
08	12.408	19.680	31.943	42.097	48.355
16	12.495	19.945	32.388	42.457	49.633
32	12.517	20.016	32.519	42.552	49.981
64	12.523	20.034	32.553	42.577	50.079
exact	12.525	20.040	32.565	42.585	50.100

Structural Damping

For a general MDOF system the damping is represented by the damping matrix C. The damping can be split into a *distributed damping C_s* and *concentrated nodal or element damping C_n*:

$$\mathbf{C} = \mathbf{C}_\mathrm{s} + \mathbf{C}_\mathrm{n}$$

Special damping laws must be defined for the nodal dampers. The distributed damping is, in absence of more realistic yet manageable damping models, in almost all cases captured by the so-called RAYLEIGH damping.

RAYLEIGH Damping

We assume that the damping is proportional to mass and stiffness. Then the damping depends only on two unknown free parameters α_M and α_K. We need two conditions to determined these free parameters which we find in modal space.

$$\mathbf{C} = \alpha_M \mathbf{M} + \alpha_K \mathbf{K}$$

We know from Dynamics I that both M and K become diagonal in modal space. C, being a linear combination of K and M, becomes also diagonal in modal space:

$$\tilde{c}_i = \alpha_M \tilde{m}_i + \alpha_K \tilde{k}_i = 2\xi_i \omega_i \tilde{m}_i$$

Computation of the RAYLEIGH Coefficients

Two modal damping ratios ξ_i and ξ_j for two arbitrarily chosen modes can be fixed by us to identify the parameters α_K and α_M:

mode i: (1)

$$\alpha_M + \alpha_K \omega_i^2 = 2\xi_i\omega_i$$

mode j: (2)

$$\alpha_M + \alpha_K \omega_j^2 = 2\xi_j\omega_j$$

$\omega_1 \neq \omega_2$!

$$\alpha_M = 2\frac{\xi_j\omega_j\omega_i^2 - \xi_i\omega_i\omega_j^2}{\omega_i^2 - \omega_j^2}$$

$$\alpha_K = 2\frac{\xi_i\omega_i - \xi_j\omega_j}{\omega_i^2 - \omega_j^2}$$

Special Cases

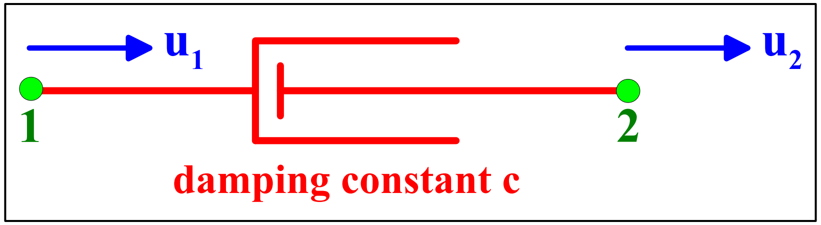
Case A: stiffness-proportional damping

$$\alpha_K = 2\frac{\xi}{\omega}$$

Case B: mass-proportional damping

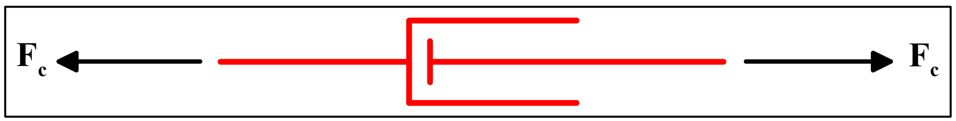
$$\alpha_M = 2\xi\omega$$

Damping Element



The damping element has two nodes with a dashpot in between. Its viscous damping properties are described by a damping constant c (not by a damping coefficient ξ), measured in [kNs/m].

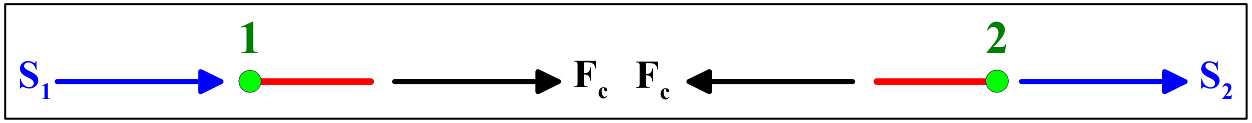
damping force



damping law

$$F_c = c \cdot \Delta\dot{u} = c \cdot (\dot{u}_2 - \dot{u}_1)$$

nodal equilibrium



element damping matrix

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} -F_c \\ F_c \end{bmatrix} = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \rightarrow \mathbf{s} = \mathbf{c} \cdot \dot{\mathbf{v}}$$

Flow Chart for Introduction of Damping in an FE-Analsyis

discretize structure

eigenfrequency analysis: ω_i, Φ_i

visualise modes Φ_i

choose 2 modes i, j with ξ_i and ξ_j

calculate α_M and α_K

perform TD FE-analysis with α_M and α_K

Summary

Inertial effects are captured in the element mass matrix. There are two alternative formulations:

The consistent mass matrix. Here we use the same shape functions for the discretization of the work of the inertial forces as we used for the stiffness matrix. We get a fully populated element matrix where each degree of freedom, also rotations, is given some mass contribution. The storage image on the system level is identical to the one of the system stiffness matrix.

The lumped mass matrix. Here we lump the element mass into nodal masses at each node. The storage requirement is greatly reduced, but it is not always evident how to distribute the total element mass to the nodes for more complex element types with many nodes.

Both formulations must converge to the true solution in the case of more and more refined element meshes. One good test of a numerical model therefore consists in running both alternatives and comparing the results. Larger discrepancies indicate some defect in our modelling.

Damping is usually captured for direct time integration via mass- and stiffness-proportional damping: *RAYLEIGH damping*. Here we can control the damping ratios of two modes – the other modes are automatically damped. In addition to the global structural damping via the RAYLEIGH concept we can also introduce *discrete dampers*. The damping matrix of a *viscous damper* is identical to the stiffness matrix of a couple spring where we replace the spring stiffness with the damping constant.