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Lecture Series:

Structural Dynamics

Lecture 9:

Frequency Domain Approach

Overview

- Transformation of the solution domain
- Complex solution of the SDOF oscillator under harmonic loads
- Frequency Domain Approach:
- Example

Transformation of a Problem: Integration

Mathematical problem: integration of a non-trivial function

$$I = \int \frac{1}{\sqrt{1-x^2}}\,dx = ?$$

The solution is not obvious!

Transformation of the problem:
Substitution of x by a suitable coordinate transformation

$$x = \sin z \quad \Rightarrow \quad dx = \cos z \, dz$$

Solution of the Transformed Problem and Re-Transformation

$$I = \int \frac{1}{\sqrt{1-(\sin z)^2}} \cos z \, dz = \int dz = z$$

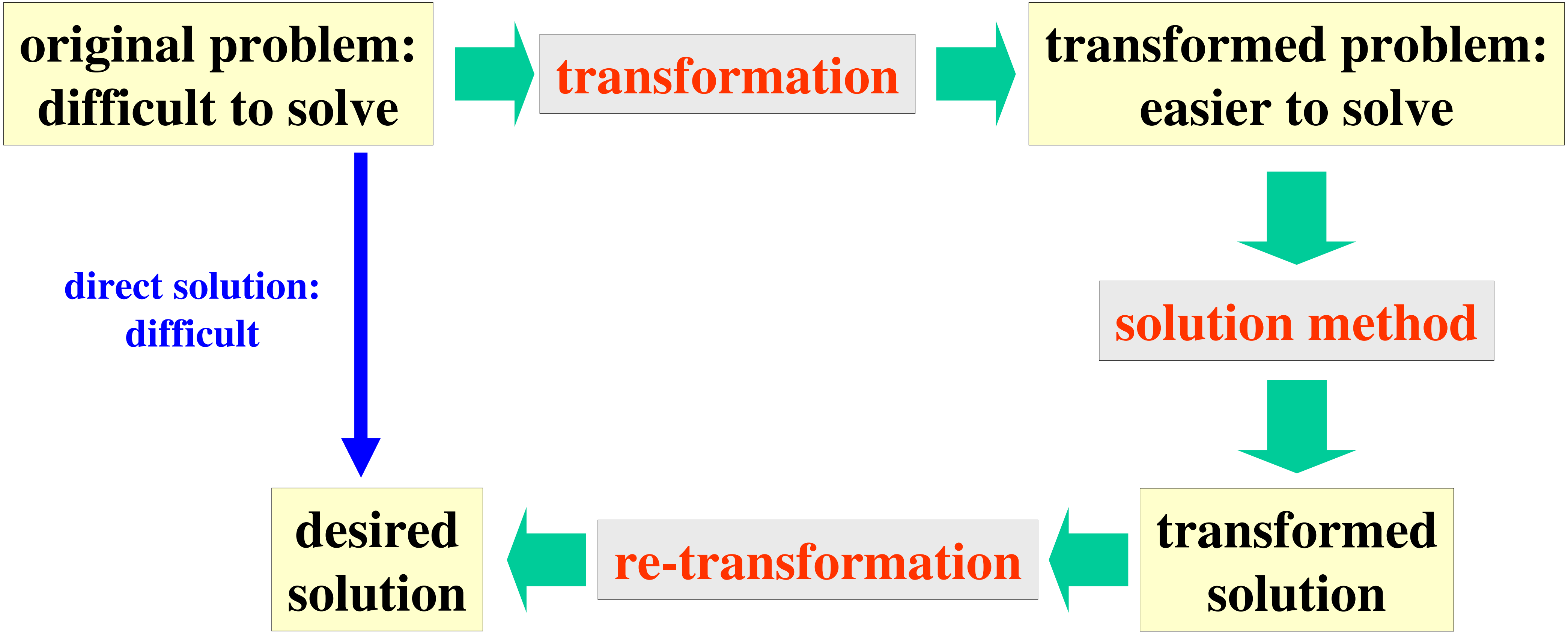
The integration is now trivial!

re-substitution

$$I = \arcsin(x)$$

The original (difficult) problem became much easier to solve by transforming it into a more suitable *solution domain*. A re-transformation then yields the desired solution. This technique is not restricted to integration but can be applied to any problem.

Generalisation



Outlook

We have already encountered several methods to calculate the response of SDOF and MDOF systems to different types of loading, e.g. the ***DUHAMEL integral***, the ***mode superposition*** technique or ***direct time integration***. All these techniques have one thing in common: they work in the ***time domain***. That mean that we solve the equation in such a way that we obtain immediately the response as a ***function of time***.

An alternative to the time domain approach is the so-called ***frequency domain*** approach. This consists in transforming the problem into another solution domain, viz. the frequency domain where we obtain the response not as a function of time but as a ***function of frequency***. We then transform the frequency domain solution back into the time domain to obtain the 'usual' time domain response. The frequency domain approach allows us to capture some effects which cannot be modelled in the time domain.

The first step in constructing this method consists in formulating harmonic vibrations as complex vibrations. Let's see what we mean by that.

Representation of Harmonic Oscillations in the Complex Realm

An oscillating structure, e.g. a mast, is a phenomenon in nature. We can observe the vibration with our naked eye simply by looking at it. If we measure it, we obtain a time function v(t) which assigns a specific value v to each time instance t. These values v are always *real numbers*, e.g. v = 2.3 mm. There is no imaginary part of v: what would be the meaning of v = (3+4i) mm?

Of special interest is a harmonic oscillation in general form: it is given by a sine with a circular frequency ω and with a phase ψ:

$$v(t) = \hat{v}\sin(\omega t + \psi)$$

Now we remember that any complex number can be expressed in the GAUSSIAN number plane by its amplitude r and its phase φ:

$$\underline{z} = r(\cos\varphi + i\sin\varphi)$$

We observe a certain correspondence between the physical oscillation and the imaginary part of a complex number: both are described by a sine function.

Now we define the ***complex oscillation*** $\underline{v}$ such that the imaginary part is identical to the physical oscillation in nature:

$$\underline{v}(t) = \hat{v}\,(\cos(\omega t + \psi) + i\,\sin(\omega t + \psi))$$

We use EULER's formula to express the complex oscillation in exponential form:

$$\underline{v}(t) = \hat{v}\,e^{i(\omega t + \psi)} = \hat{v}\,e^{i\psi}\,e^{i\omega t}$$

The exponential complex form allows us to split phase ψ and frequency ω into separate multiplicative terms. Such a separation is not possible for the original real sine function. The product of the two terms for the amplitude and the phase can be interpreted as a complex number in their own right, the ***complex amplitude***. The complex amplitude subsumes both the physical amplitude, e.g. in [mm], and the phase angle.

$$\underline{\hat{v}} = \hat{v}\,e^{i\psi} = \hat{v}\,(\cos\psi + i\,\sin\psi) \quad \Rightarrow \quad \underline{v}(t) = \underline{\hat{v}}\,e^{i\omega t}$$

Extraction of the „True“ Oscillation

The real oscillation is the imaginary part of the complex oscillation:

$$v(t) = \mathrm{Im}(\underline{v}(t))$$

The real amplitude and phase result from the absolute value and phase of the complex oscillation:

$$\hat{v} = \left|\underline{\hat{v}}\right| = \sqrt{\mathrm{Re}^2(\underline{\hat{v}}) + \mathrm{Im}^2(\underline{\hat{v}})}$$

$$\tan\psi = \frac{\mathrm{Im}(\underline{\hat{v}})}{\mathrm{Re}(\underline{\hat{v}})}$$

Where does the complex oscillation come from? In the next step we will demonstrate that it is possible to solve differential equations in the complex realm which results in the solutions being complex functions.

Forced Vibration under Harmonic Loading

Harmonic loading was the single most important load type since it produced the effect of resonance. Periodic loading could be expressed as a superposition of evenly spaced harmonics. In this lecture we will see that also transient responses can be expressed by harmonics via a FOURIER integral. To achieve this insight we have to solve the SDOF oscillator under harmonic loading in the complex realm. We recapitulate the time domain solution: a harmonic load with load frequency Ω produces a response with the same frequency which is shifted by the phase angle φ. The displacement amplitude can be expressed by the static solution amplified by the dynamic amplification V_1.

harmonic load:

$$p(t) = \hat{p}\sin(\Omega t + \Psi)$$

equation of motion:

$$m\ddot{v} + c\dot{v} + kv = \hat{p}\sin(\Omega t + \Psi)$$

solution

$$v(t) = \hat{v}\sin(\Omega t + \psi - \varphi)$$

dynamic amplification:

$$\hat{v} = v_{stat}\frac{1}{\sqrt{(2\xi\eta)^2 + (1-\eta^2)^2}} = V_1 \cdot v_{stat}$$

phase angle:

$$\tan\varphi = \frac{2\xi\eta}{1-\eta^2}$$

Complex Differential Equation of Motion

real load:

$$p(t) = \hat{p}\sin(\Omega t + \psi)$$

complex load:

$$\underline{p}(t) = \hat{p}\,e^{i(\Omega t + \Psi)} = \hat{\underline{p}}\,e^{i\Omega t}$$

complex load amplitude:

$$\hat{\underline{p}} = \hat{p}\,e^{i\Psi}$$

complex equation of motion:

$$m\,\ddot{\underline{v}} + c\,\dot{\underline{v}} + k\,\underline{v} = \hat{\underline{p}}\,e^{i\Omega t}$$

We extend the true real load by adding a fictitious cosine part and integrating the sine load in the imaginary part. The load phase ψ can be extracted into the complex load amplitude so that the time-dependent part of p(t) contains only a complex harmonic function without phase.

Trial Function for Displacements

We choose a trial function which has the same frequency as the load but a different phase φ:

displacements:

$$\underline{v}(t) = \hat{v}\, e^{i(\Omega t + \varphi)} = \underline{\hat{v}}\, e^{i\Omega t}$$

velocities:

$$\underline{\dot{v}}(t) = i\,\Omega\,\underline{\hat{v}}\, e^{i\Omega t}$$

accelerations:

$$\underline{\ddot{v}}(t) = -\Omega^2\,\underline{\hat{v}}\, e^{i\Omega t}$$

complex displacement amplitude:

$$\underline{\hat{v}} = \hat{v}\, e^{i\varphi}$$

Solution of the Equation of Motion

complex equation of motion:

$$m\,\ddot{\underline{v}} + c\,\dot{\underline{v}} + k\,\underline{v} = \underline{\hat{p}}\,e^{i\Omega t}$$

substitute trial functions

$$-m\Omega^2\,\underline{\hat{v}}\,e^{i\Omega t} + c\,i\,\Omega\,\underline{\hat{v}}\,e^{i\Omega t} + k\,\underline{\hat{v}}\,e^{i\Omega t} = \underline{\hat{p}}\,e^{i\Omega t}$$

divide by the frequency term

$$\{k - m\Omega^2 + ic\Omega\}\,\underline{\hat{v}} = \underline{\hat{p}}$$

solve for complex displacement amplitude

$$\underline{\hat{v}} = \underline{\hat{p}}\,\frac{1}{k - m\Omega^2 + ic\Omega}$$

Transfer Function

Elimination of the imaginary part in the denominator:

$$\underline{\hat{v}} = \underline{\hat{p}} \, \frac{k - m\Omega^2 - ic\Omega}{(k - m\Omega^2)^2 + (c\Omega)^2}$$

$$\underline{\hat{v}} = \underline{H}(i\Omega)\,\underline{\hat{p}}$$

H is called *transfer function*. A transfer function relates the *complex amplitude* of an *input process* to the complex amplitude of an *output process*. The system, in our case the oscillator, changes both *amplitude* and *phase* of the input process. The transfer function, being complex, contains the entire information in one single function. It is a function of the *frequency*: harmonics of different frequencies are differently amplified and phase-shifted. A transfer function contains the entire properties of the system in question. In our case these properties regard the mechanical properties of the oscillator: we speak of the *mechanical transfer function*. Transfer functions are a fundamental aspect of dynamic systems.

Transfer Function with Frequency Ratio

In Lecture 5 we introduced the *frequency ratio η* to demonstrate that the *dynamic amplification* depends only on η and the damping ξ. We do the same here. H then also depends only on η and ξ; the stiffness k serves at transforming the unit of the load amplitude, i.e. force, into the unit of the displacement, i.e. length. It must be possible to extract the dynamic amplification V_1 and the phase shift φ from the complex transfer function. For this we need the time domain response.

frequency ratio:

$$\eta = \frac{\Omega}{\omega}$$

transfer function:

$$\underline{H}(i\Omega) = \frac{(1-\eta^2) - i\,2\xi\eta}{k\{(1-\eta^2)^2 + (2\xi\eta)^2\}}$$

Time Domain Response

trial function

$$\underline{v}(t) = \underline{\hat{v}}\, e^{i\Omega t}$$

→ **introduce transfer function** →

$$\underline{v}(t) = \underline{H}\, \underline{\hat{p}}\, e^{i\Omega t}$$

↓ **substitute load amplitude** ↓

$$\underline{v}(t) = \underline{H}\, \hat{p}\, e^{i\Psi} e^{i\Omega t}$$

← **substitute transfer function** ←

$$\underline{v}(t) = \left|\underline{H}\right| e^{i\alpha} \hat{p} e^{i\Psi} e^{i\Omega t}$$

↓ **combine exponential terms** ↓

$$\underline{v}(t) = \left|\underline{H}\right| \hat{p}\, e^{i(\Omega t+\Psi+\alpha)}$$

complex response in the time domain

Real Amplitude and Phase

Amplitude of the transfer function:

$$|\underline{H}| = \sqrt{\mathrm{Re}^2(\underline{H}) + \mathrm{Im}^2(\underline{H})}$$

$$|\underline{H}| = \frac{1}{k\sqrt{(1-\eta^2)^2 + (2\xi\eta)^2}}$$

α: phase of the complex transfer function

$$\tan\alpha = \frac{\mathrm{Im}(\underline{H})}{\mathrm{Re}(\underline{H})} = -\frac{2\xi\eta}{1-\eta^2}$$

Real Oscillation

The real oscillation is identical to the imaginary sine part of the complex solution:

$$v(t) = \frac{\hat{p}}{k} \frac{1}{\sqrt{(1-\eta^2)^2 + (2\xi\eta)^2}} \sin(\Omega t + \Psi - \varphi)$$

The sign of the phase angle φ is changed (φ = -α, tanφ = -tanα) to conform to the solution in Lecture 5:

$$\tan\varphi = +\frac{2\xi\eta}{1-\eta^2}$$

We can identify the quotient p/k as the static deformation and the factor with the root as the dynamic amplification V_1. Our solution in the complex domain yields the same real displacement as the original solution in Lecture 5!

$$v(t) = v_{stat} V_1 \sin(\Omega t + \Psi - \varphi)$$

Frequency Domain Approach: Building Blocks

Item 1: equation of motion for an arbitrary load:

$$m\,\ddot{v} + c\,\dot{v} + k\,v = p(t)$$

Item 2: solution for a complex harmonic load

$$\underline{p}(t) = \underline{\hat{p}}\,e^{i\Omega t}$$

$$\underline{\hat{v}} = \underline{H}(i\Omega)\,\underline{\hat{p}}$$

$$\underline{v}(t) = \underline{\hat{v}}\,e^{i\Omega t}$$

Item 3: complex transfer function

$$\underline{H}(i\Omega) = \frac{k - m\Omega^2 - ic\Omega}{(k - m\Omega^2)^2 + (c\Omega)^2}$$

Item 4: Fourier transformation

direct transformation

$$\underline{F}(i\Omega) = \int_{-\infty}^{+\infty} f(t)\,e^{-i\Omega t}\,dt$$

inverse transformation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{F}(i\Omega)\,e^{i\Omega t}\,d\Omega$$

Frequency Domain Approach: Derivation of the Algorithm

Step 1: FOURIER transformation of the right-hand side (load):

$$m\,\ddot{v} + c\,\dot{v} + k\,v \;=\; p(t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{P}(i\Omega)\,e^{i\Omega t}\,d\Omega$$

Now the load consists of infinite number of harmonics whose amplitudes are given by the complex FOURIER transform of p(t). The complex displacement amplitude resulting from a harmonic load can be obtained via the complex mechanical transfer function H.

Step 2: solution for a single harmonic:

$$\underline{\hat{v}}(i\Omega) \;=\; \underline{H}(i\Omega)\,\underline{P}(i\Omega) \quad\Rightarrow\quad v_{\Omega}(t) \;=\; \underline{\hat{v}}\,e^{i\Omega t}$$

Step 3: total solution by superposition:

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v_{\Omega}(t)\, d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\hat{v}}\, e^{i\Omega t}\, d\Omega$$

We assume that the system is linear, i.e. we have neither large displacements nor effects of material nonlinearity. Then the superposition principle holds: the solution for a sum of loads is identical to the sum of the individual solutions for a single load. This also holds for an integral since an integral is nothing else but an infinitely close spaced sum.

Step 4: Interpretation of the solution as a FOURIER integral

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{V}(i\Omega)\, e^{i\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\hat{v}}(i\Omega)\, e^{i\Omega t}\, d\Omega$$

The amplitude of the single solution v_{Ω} is identical to the FOURIER coefficient $\underline{V}$ for this specific frequency Ω

FD Solution

The ***FOURIER transform of the response*** can be calculated be multiplying the ***FOURIER transform of the load*** by the ***complex mechanical transfer function***. The convolution integral of the DUHAMEL integral in the TD approach the becomes a ***simple multiplication*** in the FD! The numerical effort to calculate $\underline{V}(i\Omega)$ from a given $\underline{P}(i\Omega)$ is negligible!

$$\underline{V}(i\Omega) = \underline{H}(i\Omega)\,\underline{P}(i\Omega)$$

The TD solution v(t) can then be computed by an ***inverse FOURIER transformation*** of $\underline{V}(i\Omega)$. The FD approach is a general method for any type of loading. It only requires that damping is non-zero since otherwise the transfer function would become infinite for $\Omega = \omega$ and that the system is linear.

Flow Chart for the FD Solution

time domain:

$$p(t)$$

→

direct FT:

$$\underline{P}(i\Omega) = \int_{-\infty}^{+\infty} p(t)\, e^{-i\Omega t}\, dt$$

→

frequency domain:

$$\underline{P}(i\Omega)$$

↓

transfer function:

$$\underline{V}(i\Omega) = \underline{H}(i\Omega)\, \underline{P}(i\Omega)$$

↓

frequency domain:

$$\underline{V}(i\Omega)$$

←

inverse FT

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{V}(i\Omega)\, e^{i\Omega t}\, d\Omega$$

←

time domain:

$$v(t)$$

Practical Application: FFT

direct transformation

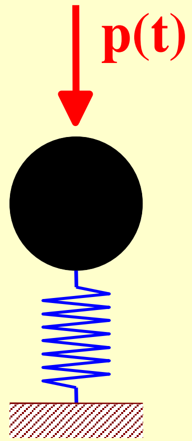
$$\underline{F}(i\Omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\Omega t}\, dt$$

inverse transformation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{F}(i\Omega)\, e^{i\Omega t}\, d\Omega$$

An exact solution via an analytical solution of the FOURIER integrals is restricted to special cases where a specific load function is given. For arbitrary loads we need a general, i.e. a numerical approach. This approach assumes that the load is given by a discrete sample of pairs (t,p). This sample can be transformed into the FD via the FFT so that we obtain a discrete FOURIER transform whose values are only correct up to the NYQUIST frequency. For these discrete frequencies we compute the transfer function H and obtain the discrete FOURIER transform of the response by a simple multiplication. An inverse FFT yields the time domain response. The IFFT requires FOURIER coefficients for the entire frequency range, i.e. also frequencies beyond the NYQUIST frequency up to the maximum frequency f_{max}. We compute these values by mirroring with respect to the NYQUIST frequency. We illustrate the process by a simple example.

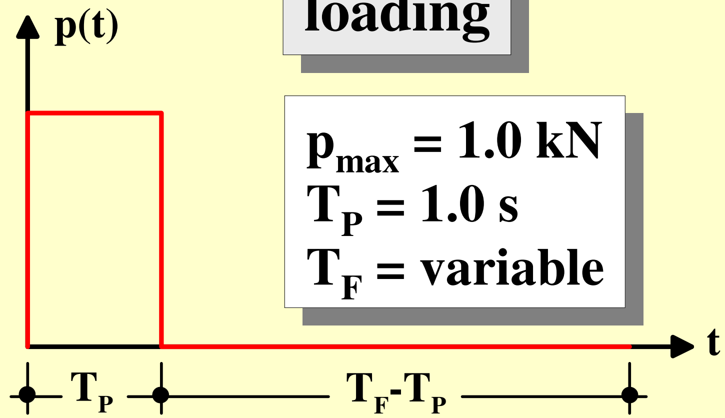
Example: Rectangular Impulse



structure

k = 1000.0 kN/m
m = 1.0 ton
ξ = 2.0 %

loading



p_{max} = 1.0 kN
T_P = 1.0 s
T_F = variable

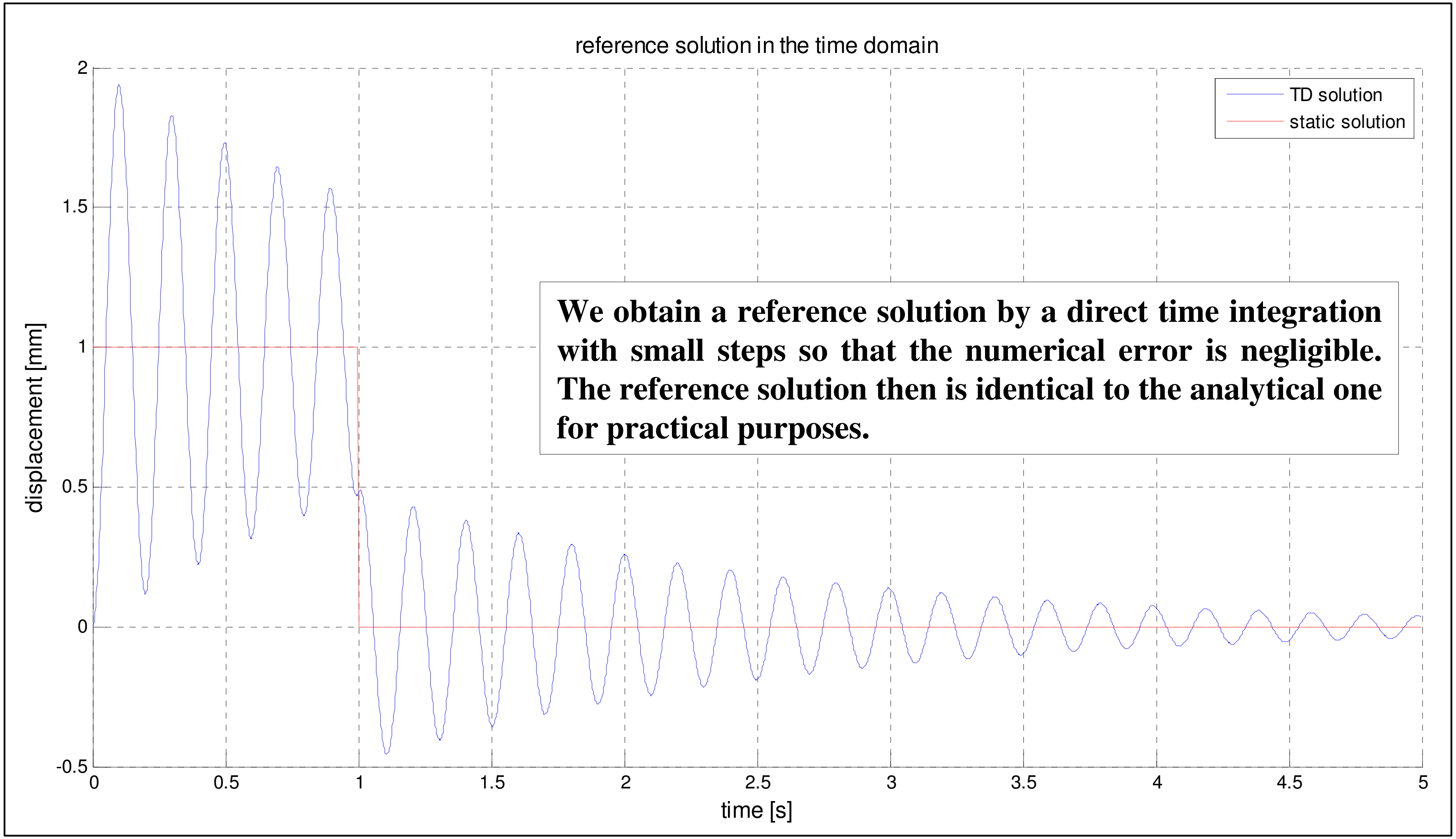
dynamic properties:

$$\omega = \sqrt{\frac{1000.0}{1.0}} = 31.62\,\text{rad/s}$$

$$f = \frac{\omega}{2\pi} = 5.03\,\text{Hz}$$

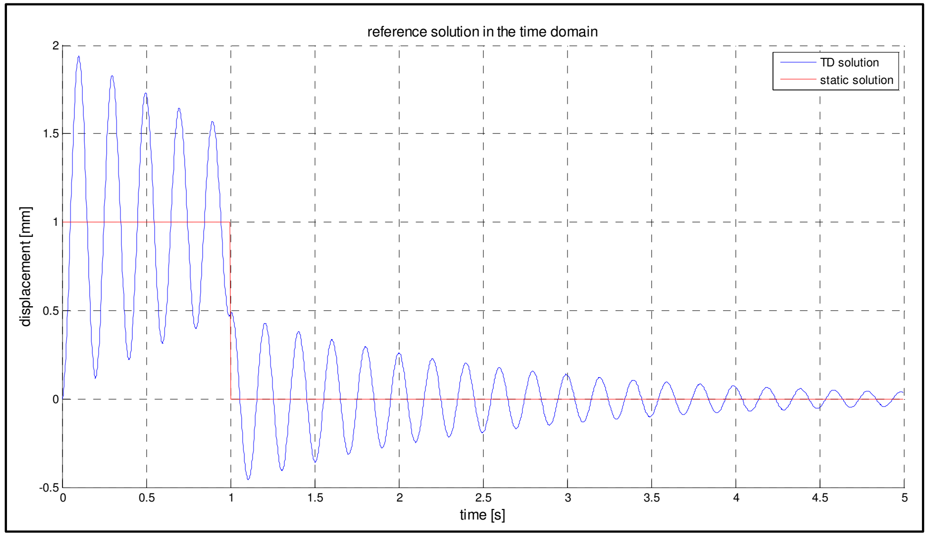
$$T = \frac{1}{f} = 0.20\,\text{s}$$

Reference Solution



We obtain a reference solution by a direct time integration with small steps so that the numerical error is negligible. The reference solution then is identical to the analytical one for practical purposes.

Dynamic Behaviour



$$f = \frac{\omega}{2\pi} = 5.03\,\text{Hz}$$

$$T = 0.20\,\text{s}$$

$$T_P = 1.00\,\text{s}$$

$$\frac{T_P}{T} = \frac{1.0}{0.2} = 5$$

We have a transient impulsive loading. The load period T_P is long with respect to the structural period so that the response contains a noticeable static part: we have a damped oscillation with the eigenfrequency of 5 Hz about the static deformation. The static deformation is 1.0 mm for the duration of the load impulse and zero afterwards. The dynamically amplified part of the vibration is dominated by a harmonic with 5 Hz.

First we take only 2^4 = 16 steps to illustrate the numerical procedure. For an arbitrarily chosen T_F of 10 s we get a time step of Δt = 10/15 = 0.67 s which results in a sampling frequency of f_{sample} = 1.5 Hz, a mirror NYQUIST frequency of f_{ny} = 0.8 Hz and a frequency resolution of 0.1 Hz. It is clear that we will not be able to capture the dominant response frequency with this sampling frequency – we are only interested in studying the procedure per se.

FD Algorithm: FFT of the Load

P(t)	
t [s]	p
0.000000	1.00
0.666667	1.00
1.333333	0.00
2.000000	0.00
2.666667	0.00
3.333333	0.00
4.000000	0.00
4.666667	0.00
5.333333	0.00
6.000000	0.00
6.666667	0.00
7.333333	0.00
8.000000	0.00
8.666667	0.00
8.333333	0.00
10.000000	0.00

FFT →

P(f)		
F [Hz]	Re(P)	Im(P)
0.000000	0.125000	0.000000
0.100000	0.120242	-0.023918
0.200000	0.106694	-0.044194
0.300000	0.086418	-0.057742
0.400000	0.062500	-0.062500
0.500000	0.038582	-0.057742
0.600000	0.018306	-0.044194
0.700000	0.004758	-0.023918
0.800000	0.000000	0.000000
0.900000	0.004758	0.023918
1.000000	0.018306	0.044194
1.100000	0.038582	0.057742
1.200000	0.062500	0.062500
1.300000	0.086418	0.057742
1.400000	0.106694	0.044194
1.500000	0.120242	0.023918

We perform a FD analysis with 16 steps and $T_F = 10.0$ s.

FOURIER coefficients below the NYQUIST frequency, relevant data.

Conjugate complex mirror images, incorrect data.

FD Algorithm: Transfer Function

P(f)		
F [Hz]	Re(P)	Im(P)
0.000000	0.125000	0.000000
0.100000	0.120242	-0.023918
0.200000	0.106694	-0.044194
0.300000	0.086418	-0.057742
0.400000	0.062500	-0.062500
0.500000	0.038582	-0.057742
0.600000	0.018306	-0.044194
0.700000	0.004758	-0.023918
0.800000	0.000000	0.000000
0.900000	0.004758	0.023918
1.000000	0.018306	0.044194
1.100000	0.038582	0.057742
1.200000	0.062500	0.062500
1.300000	0.086418	0.057742
1.400000	0.106694	0.044194
1.500000	0.120242	0.023918

H(f)		
F [Hz]	Re(P)	Im(P)
0.000000	0.001000	0.000000
0.100000	0.001000	-0.000001
0.200000	0.001002	-0.000002
0.300000	0.001004	-0.000002
0.400000	0.001006	-0.000003
0.500000	0.001010	-0.000004
0.600000	0.001014	-0.000005
0.700000	0.001020	-0.000006
0.800000	0.001026	-0.000007
0.900000		
1.000000		
1		
1		
1		
1.400000		
1.500000		

V(f)		
F [Hz]	Re(P)	Im(P)
0.000000	0.000125	0.000000
0.100000	0.000120	-0.000024
0.200000	0.000107	-0.000044
0.300000	0.000087	-0.000058
0.400000	0.000063	-0.000063
0.500000	0.000039	-0.000058
0.600000	0.000018	-0.000045
0.700000	0.000005	-0.000024
0.800000	0.000000	0.000000
0.900000		
1.000000		
1.400000		
1.500000		

$$\underline{V}(f=0.7\ \text{Hz}) = (0.004758 - 0.023918i)(0.001020 - 0.000006i)$$
$$= 0.000005 - 0.000024i$$

FD Algorithm: IFFT of the Displacement

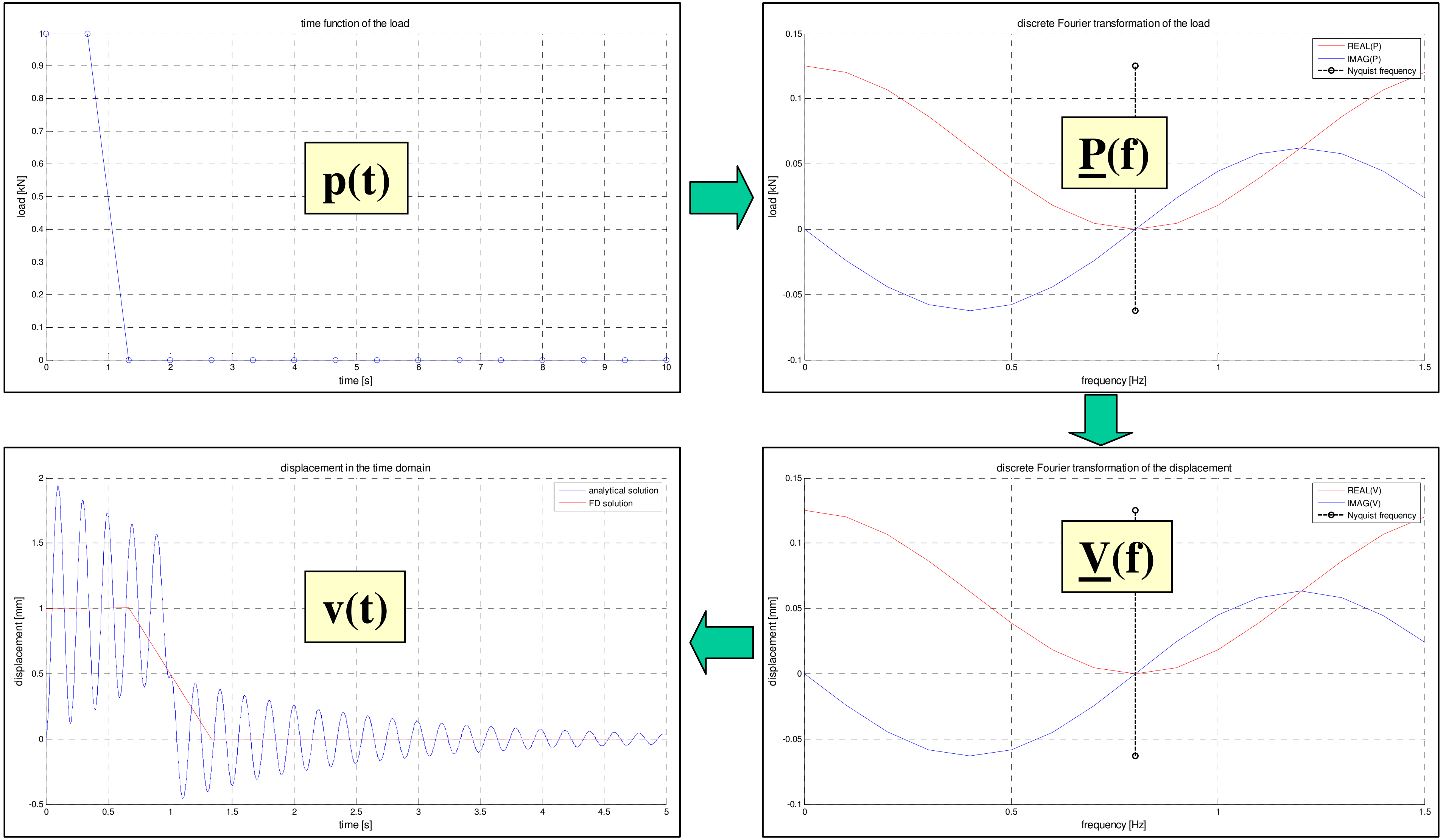
V(f)		
F [Hz]	Re(P)	Im(P)
0.000000	0.000125	0.000000
0.100000	0.000120	-0.000024
0.200000	0.000107	-0.000044
0.300000	0.000087	-0.000058
0.400000	0.000063	-0.000063
0.500000	0.000039	-0.000058
0.600000	0.000018	-0.000045
0.700000	0.000005	-0.000024
0.800000	0.000000	0.000000
0.900000		
1.000000		
1.100000		
1.200000		
1.300000		
1.400000		
1.500000		

V(f)		
F [Hz]	Re(P)	Im(P)
0.000000	0.000125	0.000000
0.100000	0.000120	-0.000024
0.200000	0.000107	-0.000044
0.300000	0.000087	-0.000058
0.400000	0.000063	-0.000063
0.500000	0.000039	-0.000058
0.600000	0.000018	-0.000045
0.700000	0.000005	-0.000024
0.800000	0.000000	0.000000
0.900000	0.000005	0.000024
1.000000	0.000018	0.000045
1.100000	0.000039	0.000058
1.200000	0.000063	0.000063
1.300000	0.000087	0.000058
1.400000	0.000107	0.000044
1.500000	0.000120	0.000024

IFFT

v(t)	
t [s]	v [mm]
0.000000	1.001285
0.666667	1.005360
1.333333	-0.002843
2.000000	0.000358
2.666667	-0.000049
3.333333	-0.000024
4.000000	0.000049
4.666667	-0.000063
5.333333	0.000075
6.000000	-0.000091
6.666667	0.000117
7.333333	-0.000163
8.000000	0.000253
8.666667	-0.000462
8.333333	0.001113
10.000000	-0.004915

Visualization of the FD Algorithm



Convergence Test

time domain → **frequency domain**

T_{sample} → $$\Delta f_{sample} = \frac{1}{T_{sample}}$$

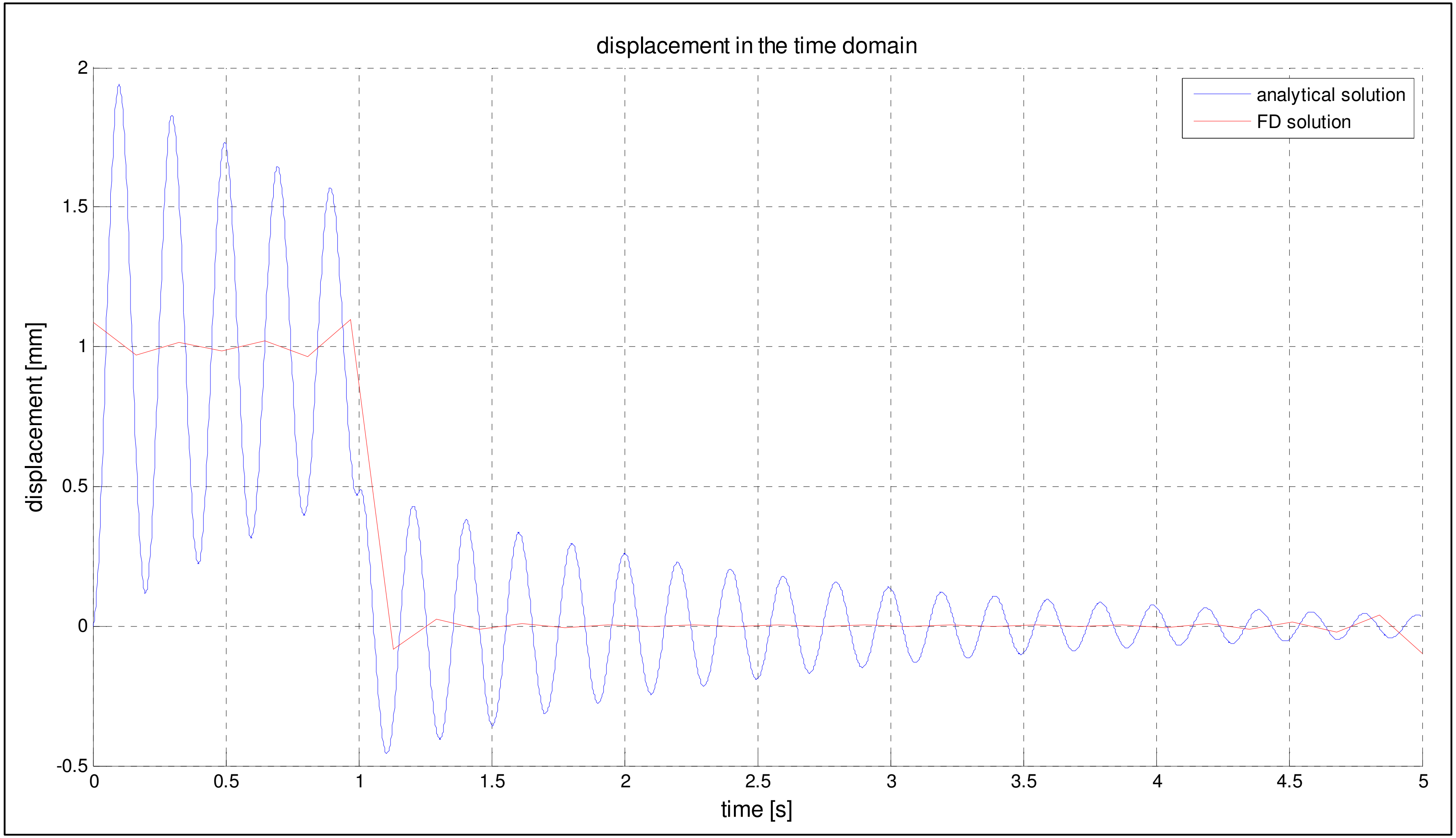
$$\Delta t_{sample} = \frac{T_{sample}}{N_{sample} - 1}$$ → $$f_{sample} = \frac{1}{\Delta t_{sample}}$$

An increase of the number of sampling points N_{sample} increases the sampling frequency f_{sample} and thereby the maximum frequency which can be captured by sampling the signal. The length of the signal T_{sample} controls the frequency resolution: a larger T_{sample} leads to a finer resolution of the frequency axis.

The best results are obtained by long samples with small time steps (i.e. a high sampling frequency). This results in large amounts of data. It does not matter as long as we treat systems with only few degrees of freedom, but it can become critical for larger problems. We test the algorithm by studying the quality of the results for varying values for T_{sample} and f_{sample}.

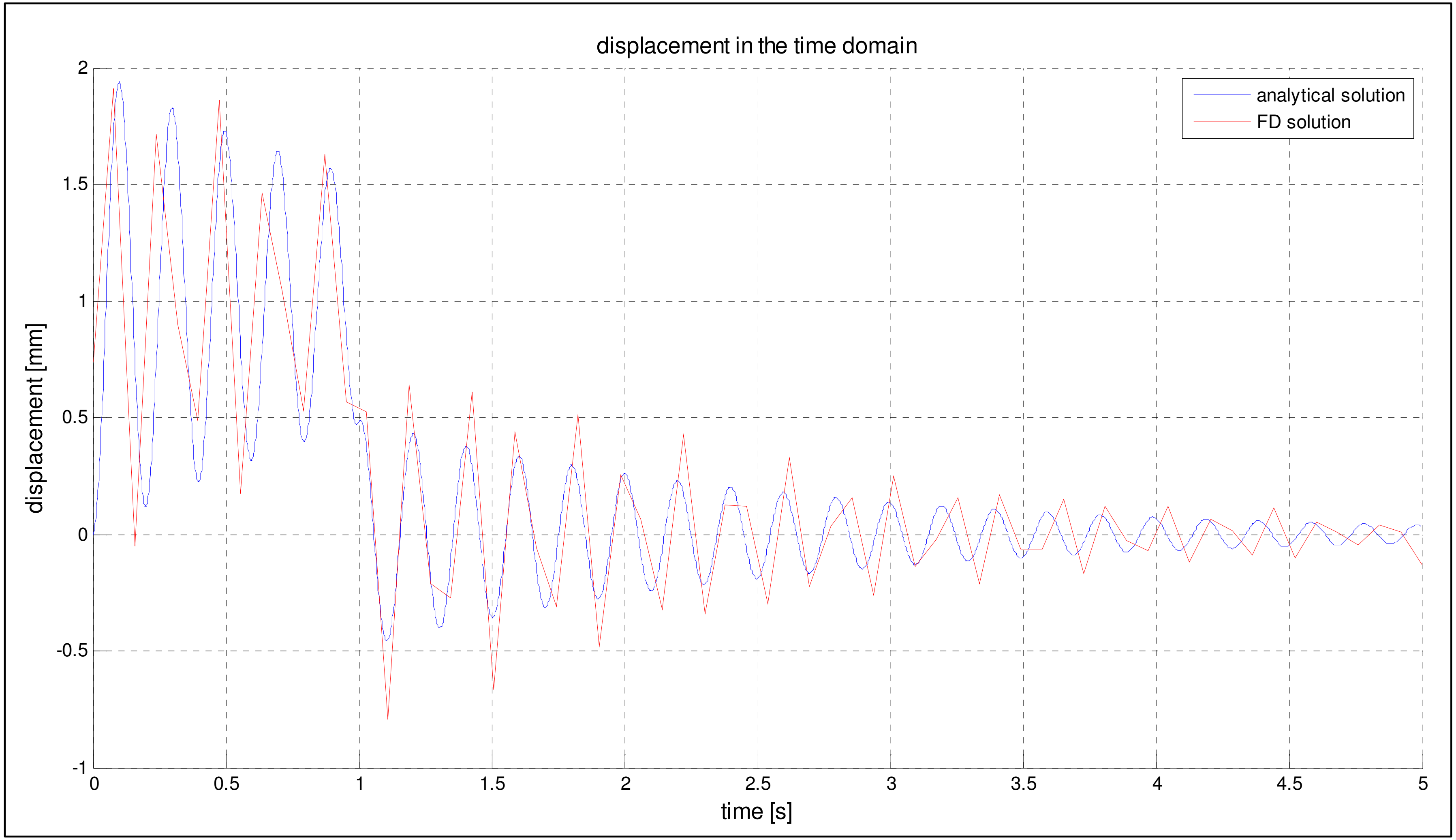
T_F = 5.0 s, Δf = 0.2 Hz, 32 Points, f_{ny} = 3.2 Hz

The dominant response frequency lies above the NYQUIST frequency. Therefore we are not able to capture the dynamics if the process and the FD solution captures only the static part of the vibration.



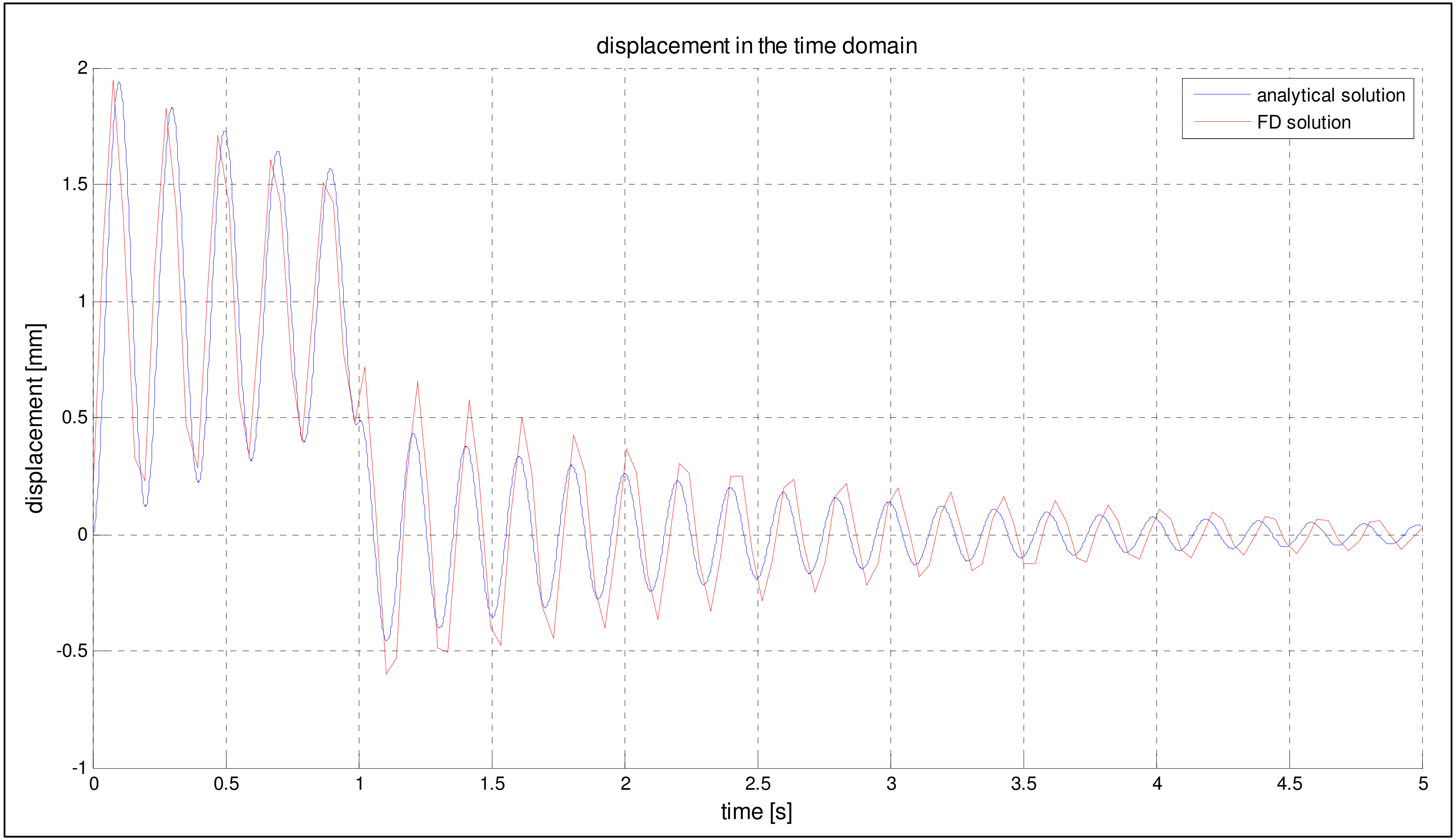
T_F = 5.0 s, Δf = 0.2 Hz, 64 Points, f_{ny} = 6.4 Hz

Now the NYQUIST frequency is high enough to capture the dominant frequency. The FD solution is qualitatively correct, yet the amplitudes show unacceptably high errors.



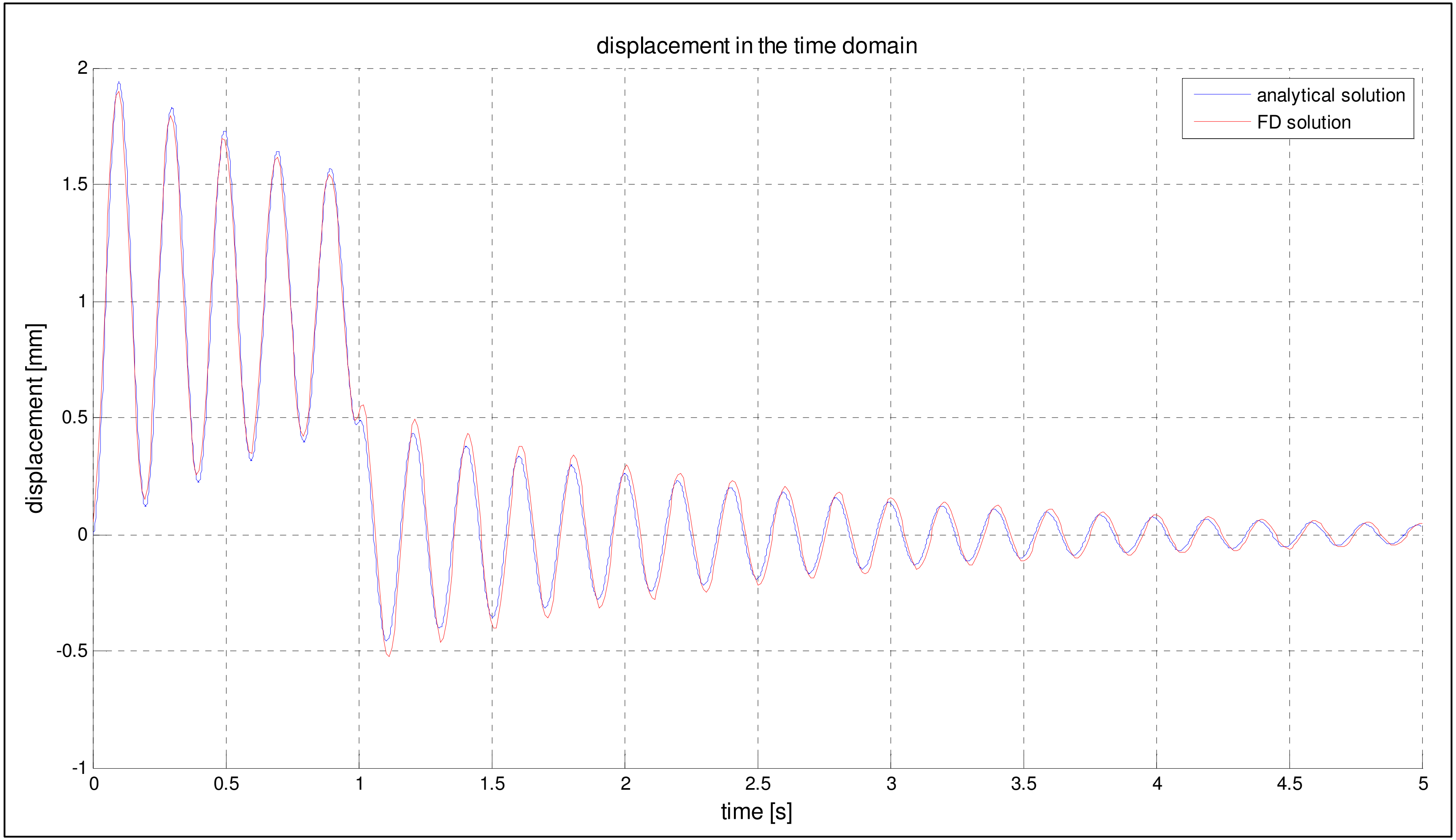
T_F = 5.0 s, Δf = 0.2 Hz, 128 Points, f_{ny} = 12.8 Hz

A further increase of the sampling frequency improves the results, in particular during the load phase. The free vibration after t = 1.0 s still shows larger deviations from the true solution.



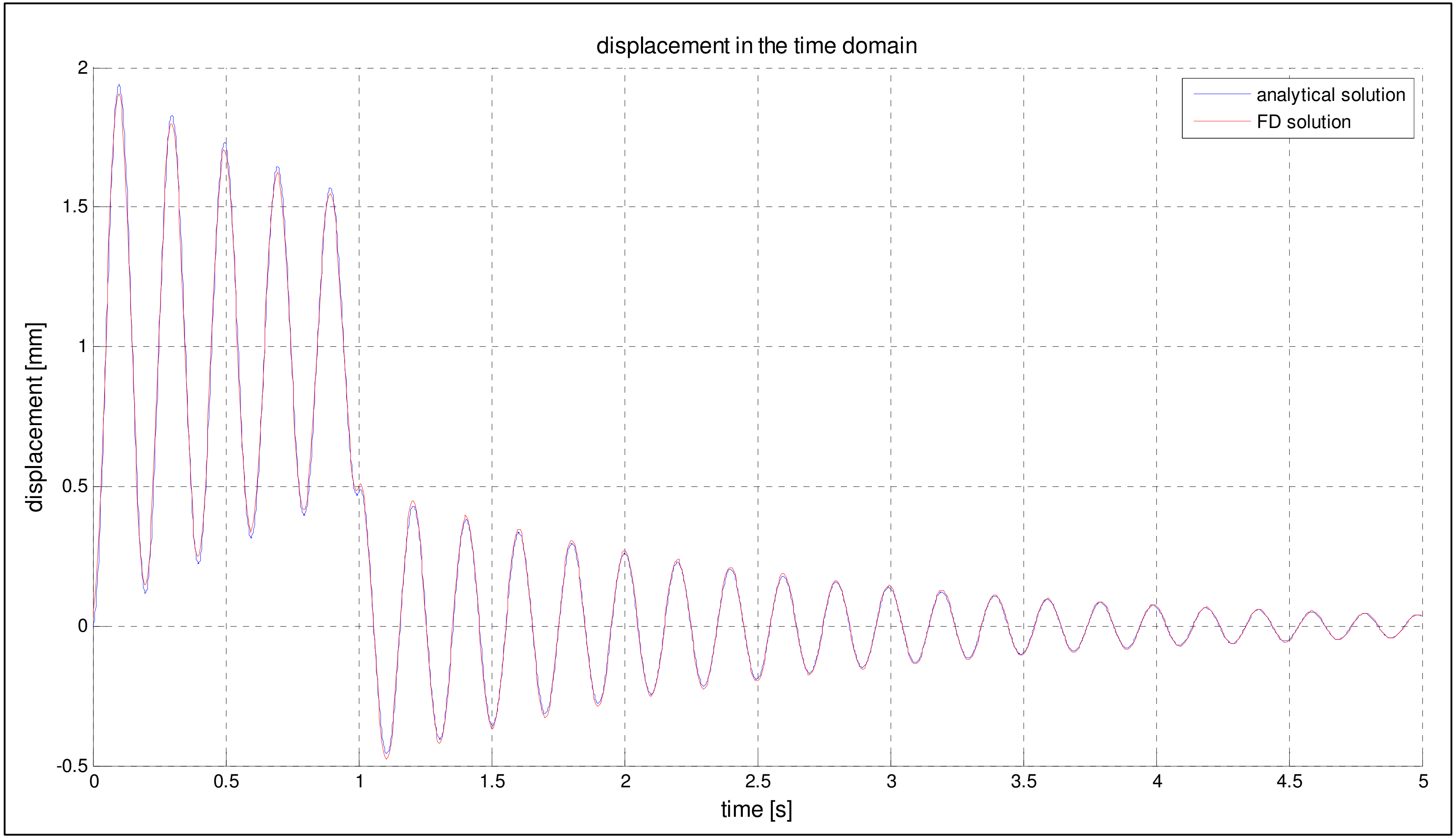
$T_F = 5.0$ s, $\Delta f = 0.2$ Hz, 512 Points, $f_{ny} = 51.2$ Hz

Now the results are good though a slight error still persists which can be perceived with the naked eye.



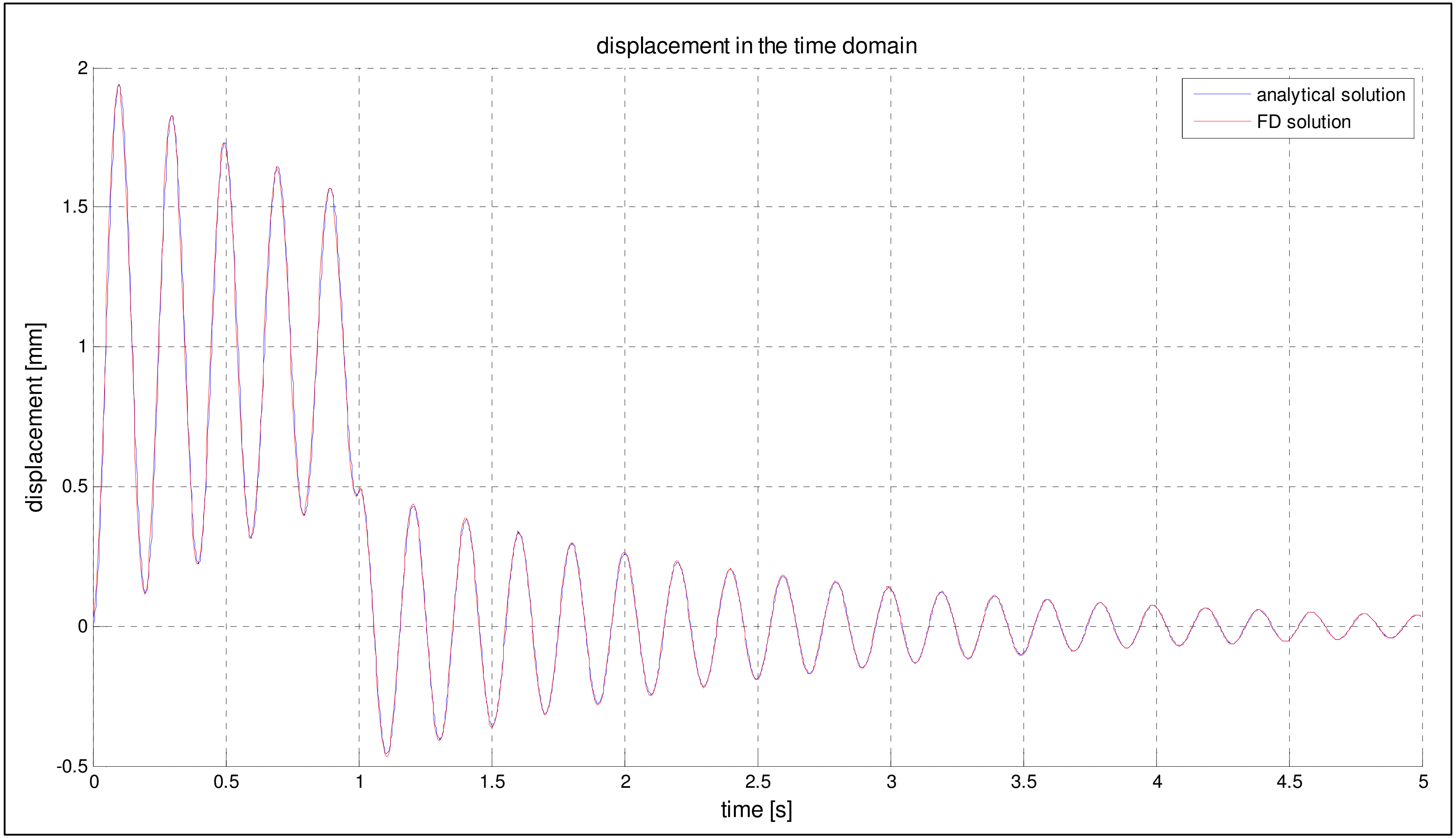
$T_F = 5.0$ s, $\Delta f = 0.2$ Hz, 1024 Points, $f_{ny} = 102.4$ Hz

We have convergence with 1024 points. The solution shows only small deviations. These, however, cannot be reduced further by further increasing the sampling frequency.



TF = 20.0 s, Δf = 0.05 Hz, 4096 Points, f_{ny} = 102.4 Hz

Instead of increasing the sampling frequency we refine the frequency resolution by quadrupling the sample length. To keep f_{sample} constant, we also have to quadruple N_{sample}. Now the FD and TD solutions are virtually identical and we can conclude that the FD approach is different but equivalent to the TD approach.



Discussion

We have seen that the frequency domain analysis yields the same time response as a time domain computation. Since we perform both methods numerically, some deviations may be observed due to numerical errors.

In the TD approach it was the size of the time step Δt which controlled the quality of the solution. We go forward from one time step to the next one, we can stop at any point, and take up the analysis from that point if further information beyond that point is needed. In a FD analysis the situation is more involved. We have *two parameters* to observe:

The *total length T_{tot}* of the analysis has to be fixed beforehand. We get the total response from the IFFT and it is not possible to obtain the response beyond that point, unless we repeat the analysis from scratch with a larger T_{tot}. The choice of T_{tot} does not only fix the time range of the response, it also fixes the frequency resolution which in turn influences the quality of the results so that T_{tot} influence the response also for time before T_{tot}.

The *time step Δt* controls the maximum frequency which can be captured. We have seen that Δt must be chosen such that the corresponding NYQUIST frequency lies beyond the maximum relevant frequency which is determined by the highest mode expected to be excited and the highest load frequency to be represented.

The FD approach not only represents an alternative to the TD analysis, it also allows us to treat problems which cannot be dealt with in the TD: *frequency-dependent structural properties*.

Frequency-Dependent Structural Properties

Our problems have hitherto one thing in common: the structural properties mass, damping and stiffness were fixed values. In particular did the structural properties not depend on the frequency of the response. That seemed natural to us and we did not question the fact that we could use the numerical model, e.g. an FE-model, from the static analysis also for the dynamic simulation. There are cases, however, where that is not correct, where properties do depend on the frequency.

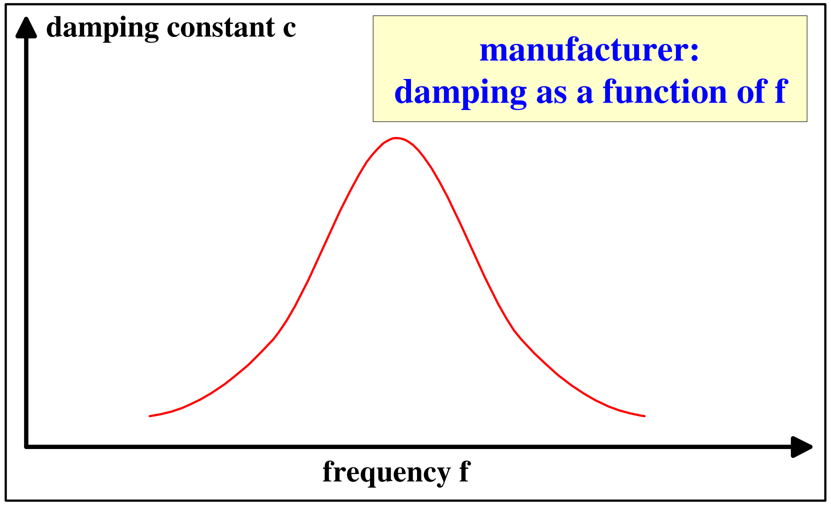
Frequency-dependent damping. A liquid damper, e.g., acts as a damping device because the liquid within it is excited to sloshing waves which dissipate energy by their motion. The forming of waves, the size of the wave amplitudes, and the question whether the waves break or not depends on the ratio of the eigenfrequencies of the liquid body and the frequency of the motion of the damper. If the damper oscillates with very low or very high frequency, only small waves are formed and the damping is low.

Frequency-dependent stiffness. A frequency-dependent stiffness might be encountered in geotechnics. Soil is an aggregate of different constituents with pores and water whose combination yields the macroscopically observable deformability, i.e. the resulting stiffness. The complex relationship between these constituents results in a soil stiffness which depends on the frequency of the soil vibration.

How to Capture Frequency-Dependent Properties

In a TD approach it is, as a matter of principle, not possible to capture frequency-dependent structural properties – the frequency of the response does not enter anywhere into the analysis. We might later, after the results have been computed, find the frequency distribution of the response by an FFT of the time history but that is just post-processing.

It is the nature of the FD approach to be formulated in terms of frequencies. The transfer function H describes the frequency-dependent flexibility, where the dependency on the frequency is caused by mass and damping effects. If also damping or stiffness are given as functions of f or Ω, then we would simply introduce these functions into H, as shown below. Each harmonic in the response would then be calculated with its own stiffness or damping.



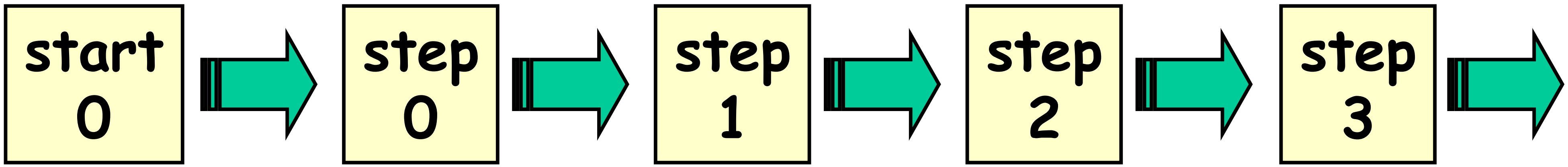
frequency-dependent transfer function

$$\underline{V}(i\Omega) = \underline{H}(i\Omega)\,\underline{P}(i\Omega)$$

$$\underline{H}(i\Omega) = \frac{k(\Omega) - m\Omega^2 - i \cdot c(\Omega) \cdot \Omega}{(k(\Omega) - m\Omega^2)^2 + (c(\Omega) \cdot \Omega)^2}$$

Storage Requirements I: Time-Stepping Methods

time-stepping procedure



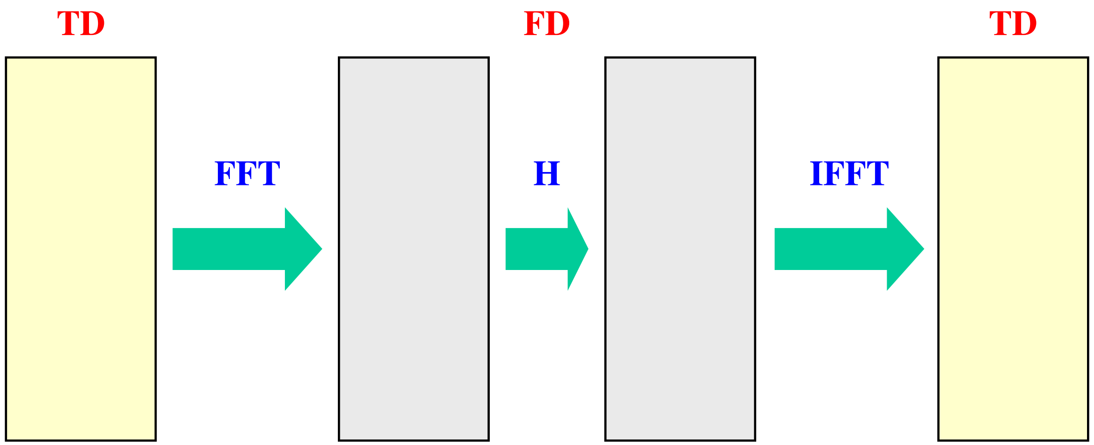
In a time stepping procedure we need basically three sets of data:

- **structural data: K, M, C, K_{eff},**
- **"now" data: P_{eff}, V_{now}**
- **"old" data: V_{old}, $\dot{V}_{old}$, $\ddot{V}_{old}$**

The "old" data comprise n previous steps for an n-step algorithm. The most widely-used NEWMARK algorithm is a 1-step algorithm. All results farther back than one step can be safely forgotten by the program once they have been written to disc. The storage requirement does therefore not depend on the length of the analysis. For an SDOF problem NEWMARK would require just 9 values.

Storage Requirements II: FD Method

FD method



The FFT or IFFT algorithms require the entire signal as argument and yield the entire transformed signal as output argument. Even if we by optimized programming re-use the same storage segment by overwriting obsolete information, we would still need one entire complex signal. It would require 2·N_{step} values (N_{step} = 1024, 2048, etc). Thus the storage requirement depends on the length of the analysis. For a system with N_{dof} degrees of freedom we would have not a single transfer function but a fully populated transfer matrix with each matrix element being a function of Ω. The storage space would quickly explode and a direct modelling in frequency space would be impractical. Then a transformation into modal space would decouple the modes and we could treat each mode separately in the FD, without having to have the entire transfer matrix in RAM. The modal approach requires linearity, but that constitutes no further restriction since an FD method is restricted to linear problems anyway.

Limitation of FD Approaches

The FD approach calculates the response for each harmonic or frequency separately and synthesizes the time response by a superposition of all frequency contributions. Therefore the ***superposition principle*** must hold, i.e. the ***structure must behave linearly***!

Sources of structural nonlinearity might be ***large displacements*** (geometrical nonlinearity) or ***inelastic behaviour*** (physical nonlinearity). In these cases the proportionality between load and displacement is lost and the superposition principle does not hold. For structurally nonlinear problems we have the phenomenon that the stiffness becomes a function of the displacement history and thereby of time.

That effect can be captured in a TD approach as time appears explicitly in the time stepping algorithm. Special steps must be taken, however, to account for the nonlinearity. We will not delve further into this – that is the topic of the lecture series on "Nonlinear Problems in Structural Engineering".

An FD approach, however, does not allow us to capture nonlinear effects, which restricts its application to linear problems.