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Lecture Series:

Structural Dynamics

Lecture 7:

Mode Superposition Method

Part a: Theory

Overview

- **Orthogonality of the mode shapes**
- **Modal decomposition method:**
  - **decoupling of the equations of motion**
  - **modal damping**
  - **superposition of the mode shapes**
  - **discussion: advantages & disadvantages**

Introduction

We have up until now only developed methods for *SDOF systems*. These methods have been very useful to study *general dynamic effects* such as resonance or dynamic amplification.

Of course, the question arises in how far these results can be useful for practical problems which are usually characterized by complex (spatial) structures with many degrees of freedom, i.e. *MDOF systems*. We have already seen in the lecture on transient loading that not all existing mode shapes are excited to the same degree. Usually, only the first few (what are few? – we will discuss this later) are relevant. The SDOF model captures the true vibration state very well if only one mode is excited – then this mode is equivalent to the SDOF model. If more than one mode contributes significantly to the response, then the SDOF system captures only part of the entire response.

So we need for these cases analysis methods which are able to take into account more than one mode shape. There are basically two different approaches: the *mode superposition method* and the *direct time integration algorithms*.

We will start with mode superposition. First we will prove a mathematical property: the so-called *orthogonality of the mode shapes*. This property will enable us to transform a *system of* N_{dof} *coupled equations of motion* into N_{dof} *uncoupled equations of motion*, i.e. N_{dof} *SDOF problems*. We already know how to solve these. The total solution is then given as the sum of the individual SDOF solutions.

Review: Eigenfrequency Analysis

We remember: eigenfrequencies are always computed for the *undamped system*, since damping can be neglected for problems in civil engineering. The abstract structural model is then described by the stiffness matrix K and the mass matrix M.

Eigenvalue problem for the undamped MDOF system:

$$(\mathbf{K} - \omega^2 \mathbf{M})\,\Phi = \mathbf{0}$$

black box: eigenvalue solver

A system with N_{dof} degrees of freedom possesses N_{dof} eigenmodes. We will see that not all modes are required. So there have been developed numerical eigenvalue solvers which compute only a subset of the eigenmodes, more precisely the first N_{eig} modes with the *lowest eigenfrequencies*. We get: N_{eig} eigenvectors (free vibration modes) Φ_i plus N_{eig} corresponding eigenvalues (eigenfrequencies) ω_i with $N_{eig} \le N_{dof}$.

Orthogonality of the Eigenmodes I

Step 1: consider 2 modes Φ_i and Φ_j

We take two *different eigenmodes* i and j, e.g. mode 3 and mode 7, with eigenvectors Φ_i and Φ_j and eigenfrequencies ω_i and ω_j. Both modes satisfy the undamped homogeneous equation of motion since they both have been computed by solving the undamped eigenvalue problem. We write the two equations down:

(1) mode i:

$$(\mathbf{K} - \omega_i^2 \mathbf{M})\,\mathbf{\Phi}_i = \mathbf{0} \quad \Rightarrow \quad \mathbf{K}\,\mathbf{\Phi}_i - \omega_i^2\,\mathbf{M}\,\mathbf{\Phi}_i = \mathbf{0}$$

(2) mode j:

$$(\mathbf{K} - \omega_j^2 \mathbf{M})\,\mathbf{\Phi}_j = \mathbf{0} \quad \Rightarrow \quad \mathbf{K}\,\mathbf{\Phi}_j - \omega_j^2\,\mathbf{M}\,\mathbf{\Phi}_j = \mathbf{0}$$

Equations (1) and (2) are *vector equations*: the zero on the right-hand side is a *zero-vector* which represents the zero loading of the free vibration.

Orthogonality of the Eigenmodes II

Step 2: transformation into scalar equations

We now transform both equations into *scalar equations* by multiplying each with a vector. We chose the eigenvector Φ_j for the equation of mode i and vice versa. Now the zero on the right-hand side is no longer a vector but the scalar value of zero.

Multiply with the transpose of the other eigenmode

$$\Phi_j^T \, \mathbf{K} \, \Phi_i - \omega_i^2 \, \Phi_j^T \, \mathbf{M} \, \Phi_i = 0 \qquad (1)$$

$$\Phi_i^T \, \mathbf{K} \, \Phi_j - \omega_j^2 \, \Phi_i^T \, \mathbf{M} \, \Phi_j = 0 \qquad (2)$$

Equation (2) looks very similar to equation (1): both contain products where the system matrices K and M are multiplied from left and right by the eigenvectors. The major difference is that the sequence of the multiplications is exchanged. We remedy this by *transposing* equation (2) .

Orthogonality of the Eigenmodes III

We remember from *matrix algebra*: we transpose a product of two matrices by transposing each matrix and changing the sequence of the multiplication.

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

$$(\mathbf{ABC})^T = [(\mathbf{AB})\mathbf{C}]^T = \mathbf{C}^T(\mathbf{AB})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$$

We remember from *structural mechanics*: both mass and stiffness matrices are *symmetric*. The transpose of a symmetric matrix is identical to the original matrix itself:

$$\mathbf{M}^T = \mathbf{M} \qquad \mathbf{K}^T = \mathbf{K}$$

Orthogonality of the Eigenmodes IV

Step 3: transpose equation (2)

$$\Phi_i^T \mathbf{K} \Phi_j - \omega_j^2 \Phi_i^T \mathbf{M} \Phi_j = 0 \quad \textbf{(2)}$$

$$\Phi_j^T \mathbf{K} \Phi_i - \omega_j^2 \Phi_j^T \mathbf{M} \Phi_i = 0^T = 0 \quad \textbf{(2)}^\mathbf{T}$$

Step 4: substract $(2)^T$ from (1)

$$\textbf{(1)} \quad \Phi_j^T \mathbf{K} \Phi_i - \omega_i^2 \Phi_j^T \mathbf{M} \Phi_i = 0$$

$$\textbf{(2)}^\mathbf{T} \quad \Phi_j^T \mathbf{K} \Phi_i - \omega_j^2 \Phi_j^T \mathbf{M} \Phi_i = 0$$

(1) - (2)$^\mathbf{T}$

$$(\omega_j^2 - \omega_i^2) \Phi_j^T \mathbf{M} \Phi_i = 0$$

Orthogonality of the Eigenmodes V

Assumption: The two distinct eigenmodes possess two distinct eigenfrequencies:

$$\omega_i \neq \omega_j$$

$$(\omega_j^2 - \omega_i^2)\,\mathbf{\Phi}_j^T \mathbf{M} \mathbf{\Phi}_i = 0 \quad \Rightarrow \quad \mathbf{\Phi}_j^T \mathbf{M} \mathbf{\Phi}_i = 0$$

$$\mathbf{\Phi}_j^T \mathbf{K} \mathbf{\Phi}_i - \omega_i^2 \mathbf{\Phi}_j^T \mathbf{M} \mathbf{\Phi}_i = 0 \quad \Rightarrow \quad \mathbf{\Phi}_j^T \mathbf{K} \mathbf{\Phi}_i = 0$$

$$\mathbf{i \neq j!}$$

In vector algebra two vectors are orthogonal if their scalar product is zero. That is not the case for a pair of eigenvectors. We say, however, that the eigenvectors are *orthogonal with respect to the mass and stiffness matrices*. The orthogonality with respect to a matrix is a generalization of the orthogonality of two vectors. If the matrix becomes the unity matrix, then being orthogonal with respect to the matrix is identical to being orthogonal in the geometric sense. The above properties are also valid in the case of $\omega_i = \omega_j$ which will not be proven here.

Definition of Modal Degrees of Freedom

We will use the orthogonality of the eigenvectors to *decouple the coupled differential equations of motion*. We start by introducing the so-called *modal degrees of freedom* η:

$$\mathbf{v}(t) = \eta_1(t)\cdot\mathbf{\Phi}_1 + \eta_2(t)\cdot\mathbf{\Phi}_2 + \ldots + \eta_n(t)\cdot\mathbf{\Phi}_{ndof}$$

The *total vibration state*, i.e. the vector v of the dofs, at a given time instance t can be expressed by a *superposition of mode shapes*. Each mode shape Φ_i enters with a certain weight η_i. These weights are not static but they change with time: $\eta_i = \eta_i(t)$. The mode shapes Φ, however, are supposed to be time-invariant. This holds true if neither mass matrix nor stiffness matrix change with time, in other words if the system behaves *linearly*.

Mathematically, the mode shapes act as base vectors for the displacement vector v. We know from vector algebra that any vector can be expressed in an infinity of coordinate systems where each coordinate system is defined by a set of base vectors. For instance a vector in 3D space:

$$\mathbf{v} = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z = \tilde{v}_x\tilde{\mathbf{e}}_x + \tilde{v}_y\tilde{\mathbf{e}}_y + \tilde{v}_z\tilde{\mathbf{e}}_z$$

Similar as the values v_x etc. being the components of the 3D vector are the modal degrees of freedom the (time-dependent) components of v(t) in the modal vector space.

Definition of the Modal Matrix

To achieve a more general formulation we assemble the individual modal degrees of freedom η_i into the ***vector of modal degrees of freedom η*** **and the mode shapes** ***into the modal matrix Φ*****. The modal matrix is fully populated with dimension N_{dof} x N_{dof} and exhibits no symmetries.**

$$\boldsymbol{\eta}(t) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \cdots \\ \eta_{ndof}(t) \end{bmatrix}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \ldots & \boldsymbol{\Phi}_{ndof} \end{bmatrix}$$

The modal transformation can then expressed by:

$$\mathbf{v}(t) = \boldsymbol{\Phi}\,\boldsymbol{\eta}(t) \;\Rightarrow\; \dot{\mathbf{v}}(t) = \boldsymbol{\Phi}\,\dot{\boldsymbol{\eta}}(t) \;\Rightarrow\; \ddot{\mathbf{v}}(t) = \boldsymbol{\Phi}\,\ddot{\boldsymbol{\eta}}(t)$$

We could reconstruct v(t) if η(t) were known. To compute η, we need the system of equations of motion for η: ***we must transform the original system of equations into modal space*****.**

Transformation into Modal Space – Version I

starting point:
undamped equation of motion

$$\mathbf{M}\,\ddot{\mathbf{v}} + \mathbf{K}\,\mathbf{v} = \mathbf{P}$$

substitution of modal dofs

$$\mathbf{v}(t) = \Phi\,\eta(t) \qquad \ddot{\mathbf{v}}(t) = \Phi\,\ddot{\eta}(t)$$

$$\mathbf{M}\,\Phi\,\ddot{\eta} + \mathbf{K}\,\Phi\,\eta = \mathbf{P}$$

multiplication by Φ^T

$$\Phi^T\mathbf{M}\,\Phi\,\ddot{\eta} + \Phi^T\mathbf{K}\,\Phi\,\eta = \Phi^T\mathbf{P}$$

Modal equation of motion:

$$\tilde{\mathbf{M}}\,\ddot{\eta} + \tilde{\mathbf{K}}\,\eta = \tilde{\mathbf{P}}$$

Transformation into Modal Space – Version II

starting point: principle of virtual work

$$\delta \mathbf{v}^T (\mathbf{M}\,\ddot{\mathbf{v}} + \mathbf{K}\,\mathbf{v} - \mathbf{P}) = 0$$

substitution of modal dofs

$$\mathbf{v}(t) = \Phi\,\eta(t)$$

$$\ddot{\mathbf{v}}(t) = \Phi\,\ddot{\eta}(t)$$

$$\delta \eta^T (\Phi^T \mathbf{M}\,\Phi\,\ddot{\eta} + \Phi^T \mathbf{K}\,\Phi\,\eta - \Phi^T \mathbf{P}) = 0$$

EULER-LAGRANGE equation

$$\Phi^T \mathbf{M}\,\Phi\,\ddot{\eta} + \Phi^T \mathbf{K}\,\Phi\,\eta = \Phi^T \mathbf{P}$$

Modal equation of motion:

$$\widetilde{\mathbf{M}}\,\ddot{\eta} + \widetilde{\mathbf{K}}\,\eta = \widetilde{\mathbf{P}}$$

System Matrices in Modal Space

modal mass:

$$\tilde{\mathbf{M}} = \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi}$$

modal stiffness:

$$\tilde{\mathbf{K}} = \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi}$$

modal load:

$$\tilde{\mathbf{P}} = \mathbf{\Phi}^T \mathbf{P}$$

The modal matrices are *diagonal matrices*:

$$\mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} = \begin{bmatrix} \tilde{m}_1 & 0 & \cdots & 0 \\ 0 & \tilde{m}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{m}_n \end{bmatrix}$$

$$\mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \begin{bmatrix} \tilde{k}_1 & 0 & \cdots & 0 \\ 0 & \tilde{k}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{k}_n \end{bmatrix}$$

Properties of the Modal Matrices

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$$

$$\begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \quad \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

The element in row 3 and column 2 of the matrix C results from a scalar product of the 3rd row vector of matrix A and the 2nd column vector of matrix B.

By a similar reasoning it can be shown that the ij-th element of a product of three matrices ABC is equal to the product of the i-th row vector of A with B with the j-th column vector of C.

$$\mathbf{D} = \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} \quad \Rightarrow \quad D_{ij} = \mathbf{A}_i^T \mathbf{B} \mathbf{C}_j$$

$$\Rightarrow \quad \tilde{\mathbf{M}}_{ij} = \mathbf{\Phi}_i^T \mathbf{M} \mathbf{\Phi}_j = \begin{cases} 0 & i \neq j \\ \tilde{m}_i & i = j \end{cases} \quad \Rightarrow$$

The modal mass (stiffness) matrix is diagonal!

Dynamic Problem in Modal Space

The coupled system of N_{dof} equations of motion is in modal space transformed into N_{dof} independent modal equations of motion. The original structure is now modelled as N_{dof} individual SDOF systems.

$$\tilde{m}_i \, \ddot{\eta}_i + \tilde{k}_i \, \eta_i = \tilde{p}_i$$

$\tilde{m}_i$: modal mass

$\tilde{k}_i$: modal stiffness

$\tilde{p}_i$: modal load

$$\tilde{\omega}_i = \omega_i = \sqrt{\frac{\tilde{k}_i}{\tilde{m}_i}}$$

The decoupling in modal space was caused by the *orthogonality of the eigenvectors* which in turn was a result of the solution of the *undamped eigenvalue problem*. We have therefore until now neglected the question of damping. How do we describe damping in modal space?

Modal Damping

The damping matrix in modal space would be given by:

$$\Phi^T C \Phi$$

It is, however, *not a diagonal matrix*, since the eigenvectors are *not orthogonal with respect to the damping matrix C*!

C cannot be used for modal decomposition. Instead *modal damping* is defined separately for each mode shape by choosing a *modal damping ratio* ξ_i:

$$\tilde{c}_i = \xi_i \tilde{c}_{crit,i} = 2 \xi_i \tilde{m}_i \omega_i$$

Finding the Modal Damping Ratios ξ_i

In theory:
Excite individually all relevant modal shapes and measure the logarithmic increment Λ_i. Find ξ_i by $\Lambda_i/2\pi$.

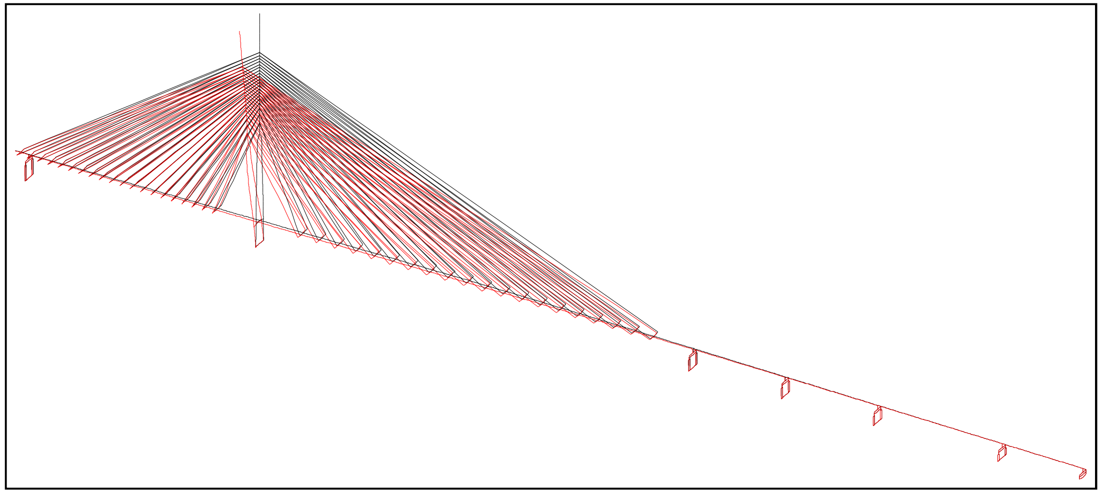
In practice:
Define damping ratios for the *fundamental mode shapes* by using tables such as in PETERSEN. Estimate the higher damping ratios from experience: The higher the mode is, the higher it is damped.

for instance:

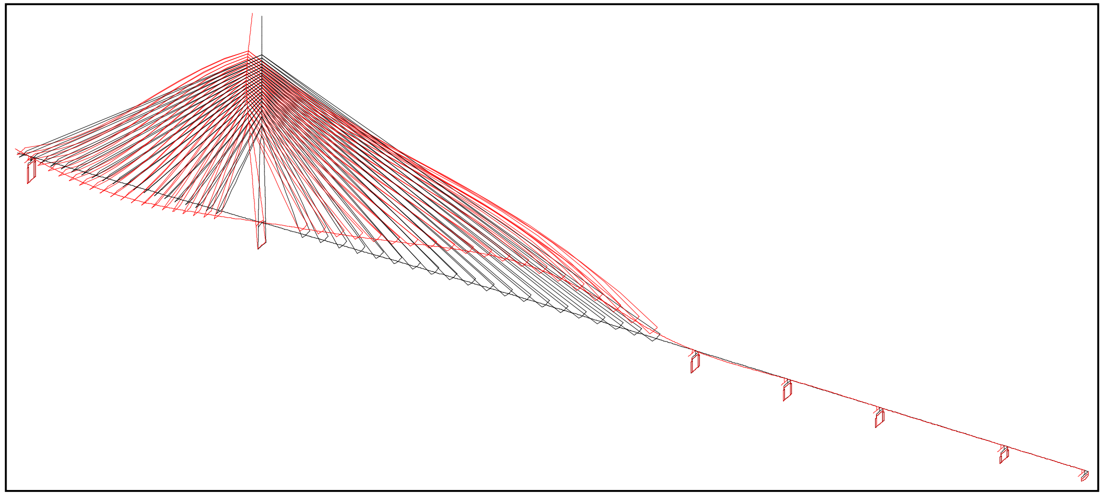
$$\xi_i = i\,\xi_1$$

Question: what are fundamental modes?

Mode Shapes of a Cable-Stayed Bridge I

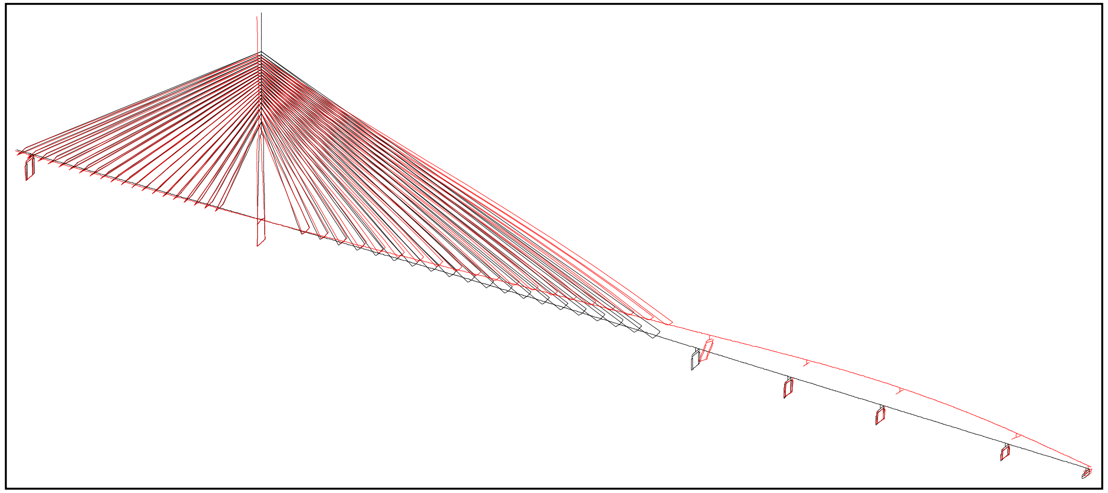


mode 1:
bending of the pylon
f_1 = 0.253 Hz

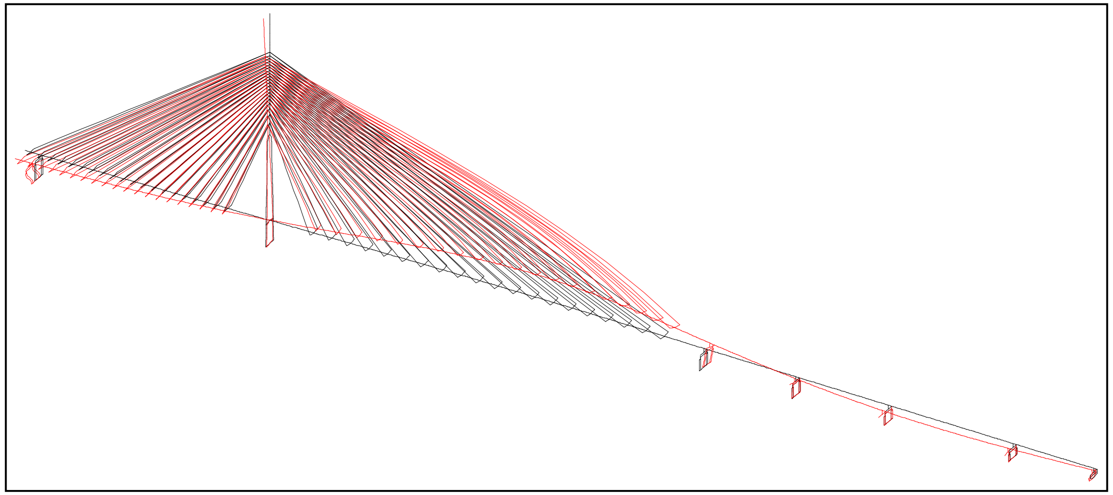


mode 2:
vertical bending of the bridge
f_2 = 0.262 Hz

Mode Shapes of a Cable-Stayed Bridge II

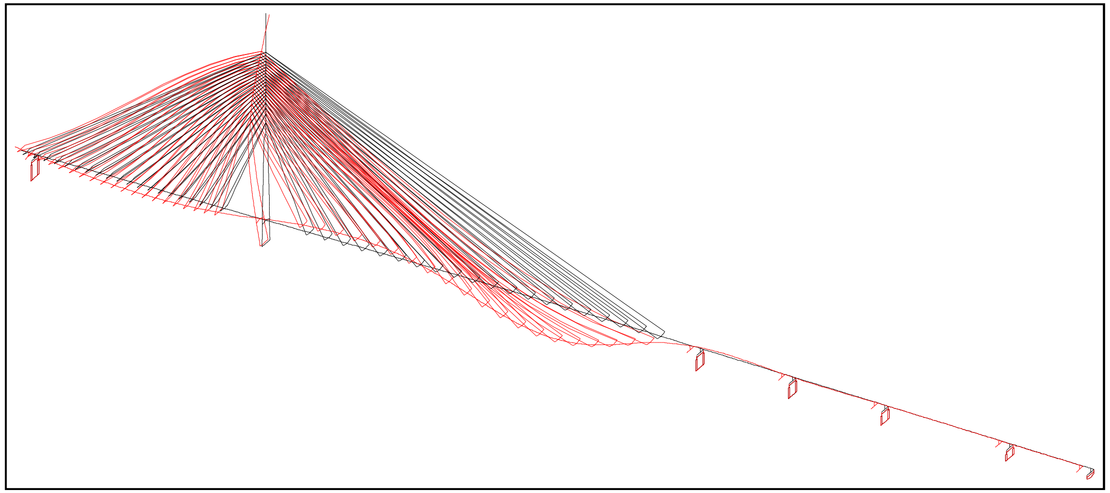


mode 3:
horizontal bending of the bridge
f_3 = 0.299 Hz

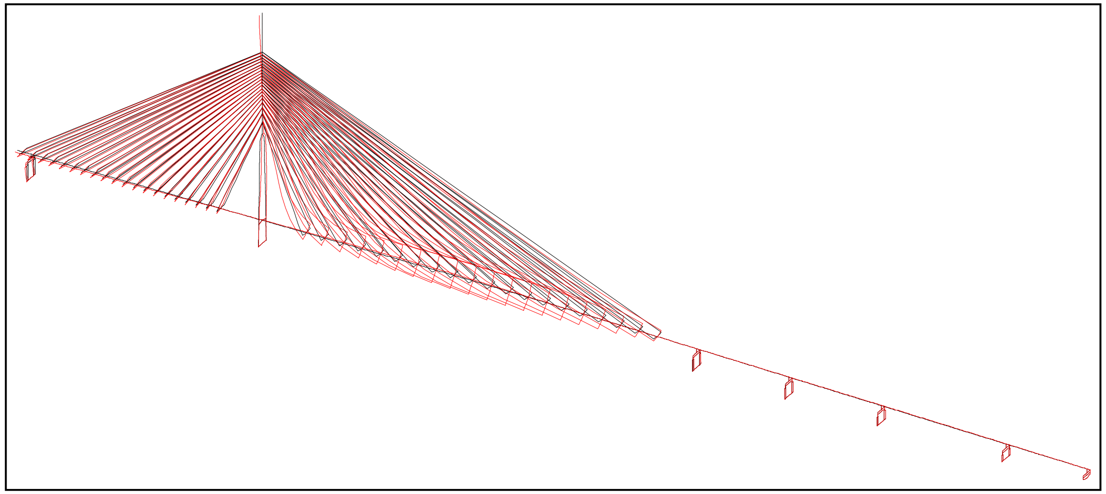


mode 4:
horizontal bending of the bridge
f_4 = 0.421 Hz

Mode Shapes of a Cable-Stayed Bridge III



mode 5:
vertical bending of the bridge
f_5 = 0.429 Hz



mode 5:
global torsion of the bridge
f_6 = 0.508 Hz

Fundamental Mode Shapes

We can see that the mode shapes of the cable-stayed bridge can be grouped into *fundamental deformation mechanisms*: *out-of-plane pylon bending*, *horizontal bending*, *vertical bending*, and *global torsion* of the bridge deck.

The basic deformation mechanisms are independent in the sense that the 1st torsional mode is a 1st mode, even if it occurs as the 6th mode in the sequence of eigenmodes. It would be nonsense to interpret it as a higher mode with respect to the 1st global mode and assign it a higher damping, e.g. 6 times the damping of the 1st out-of-plane pylon bending mode. Instead we assign modal damping according to the place of the mode within each sequence of fundamental shapes.

Proposal for the cable-stayed bridge (mainly steel – low damping):

1st horizontal bending mode: 0.5%, 2nd horizontal bending mode: 1 %, ...

1st vertical bending mode: 0.5%, 2nd vertical bending mode: 1 %, ...

1st torsional mode: 0.5%, 2nd torsional mode: 1 %, ...

1st pylon bending mode: 0.5%, 2nd pylon bending mode : 1 %, ...

It is indispensable to check the mode shapes visually before assigning modal damping!!

Synthesis of Results

Each individual modal oscillator is solved independently by any suitable solution method:

- analytical solutions (harmonic or periodic excitation)
- DUHAMEL integral
- frequency domain solution
- direct time integration
- ...

The desired solution in the original unknowns can then be constructed by a superposition of the modal responses:

$$\mathbf{v}(t) = \mathbf{\Phi}\,\boldsymbol{\eta}(t)$$

Flow Chart for an Analysis by Modal Decomposition

linear eigenfrequency analysis

$$(\mathbf{K} - \omega^2 \mathbf{M})\boldsymbol{\Phi} = \mathbf{0}$$

→

transform into modal space

$$\tilde{m}_i = \boldsymbol{\Phi}_i^T \mathbf{M} \boldsymbol{\Phi}_i , \quad \tilde{k}_i = \boldsymbol{\Phi}_i^T \mathbf{K} \boldsymbol{\Phi}_i$$

$$\tilde{p}_i = \boldsymbol{\Phi}_i^T \mathbf{P} , \quad \tilde{c}_i = 2\xi_i \tilde{m}_i \omega_i$$

↓

solve modal equations of motion

$$\tilde{m}_i \ddot{\eta}_i + 2\xi_i \tilde{m}_i \omega_i \dot{\eta}_i + \tilde{k}_i \eta_i = \tilde{p}_i$$

←

result in original space

$$\mathbf{v}(t) = \boldsymbol{\Phi} \boldsymbol{\eta}(t)$$

The mode superposition method allows us to compute the response of an MDOF system by solving a number of SDOF problems. So we do not really solve the coupled problem directly: we rely on methods developed for SDOF systems.

The true value of the modal superposition method, however, lies elsewhere. It enables us to reduce the number of degrees of freedom drastically!

Choice of Eigenvectors I

The eigenvector describes only the *shape* of a mode of vibration – its *amplitude* is *arbitrary* and is fixed by different programs in different ways. One way is to scale the length of the vector to unity, another way consists of setting one representative dof to unity. So the question arises: how do we define the eigenvectors Φ_i in modal superposition? The answer is: the final result does not depend on the choice of eigenvectors – we can fix their amplitudes arbitrarily.

We will now prove this claim by choosing an eigenvector Φ and then a multiple $\alpha \cdot \Phi$ and demonstrate that both modal shapes yield the same result.

mode shape 1

$$\mathbf{v} = \mathbf{\Phi}\eta$$

mode shape 2

$$\hat{\mathbf{v}} = \hat{\mathbf{\Phi}}\hat{\eta} = \alpha\mathbf{\Phi}\hat{\eta}$$

? →

$$\mathbf{v} = \hat{\mathbf{v}}$$

The modal equations of motion are formally identical!

modal equation of motion for mode shape 1

$$\tilde{m}\ddot{\eta} + 2\xi\tilde{m}\omega\dot{\eta} + \tilde{k}\eta = \tilde{p}$$

modal equation of motion for mode shape 2

$$\hat{m}\ddot{\hat{\eta}} + 2\xi\hat{m}\omega\dot{\hat{\eta}} + \hat{k}\hat{\eta} = \hat{p}$$

Choice of Eigenvectors II

We calculate the modal parameters for eigenvector 2 and find that the modal mass and stiffness change by α^2 and the modal load by α:

$$\hat{m} = \hat{\mathbf{\Phi}}^T \mathbf{M} \hat{\mathbf{\Phi}} = \alpha \mathbf{\Phi}^T \mathbf{M} \alpha \mathbf{\Phi} = \alpha^2 \tilde{m}$$

$$\hat{p} = \hat{\mathbf{\Phi}}^T \mathbf{P} = \alpha \mathbf{\Phi}^T \mathbf{P} = \alpha \tilde{p}$$

We substitute these parameters in the modal equation of motion:

$$\alpha^2 \tilde{m} \ddot{\hat{\eta}} + 2\xi \alpha^2 \tilde{m} \omega \dot{\hat{\eta}} + \alpha^2 \tilde{k} \hat{\eta} = \alpha \tilde{p}$$

$$\tilde{m} \ddot{\hat{\eta}} + 2\xi \tilde{m} \omega \dot{\hat{\eta}} + \tilde{k} \hat{\eta} = \frac{1}{\alpha} \tilde{p}$$

$$\hat{\eta} = \frac{1}{\alpha} \eta$$

Choice of Eigenvectors III

The modal displacements are different for eigenvector 2! If we use e.g. an eigenvector of double length, then the modal degrees of freedom are halved. Therefore it is not useful to compare modal displacements – different programs would yield different values, depending on their choice of eigenvectors. The modal displacements are no true displacements. Instead they are solutions of a *purely mathematically* derived SDOF system which has *no physical reality*.

Now we re-transform into the original solution space with the true displacement vector v:

$$\hat{\mathbf{v}} = \hat{\mathbf{\Phi}}\hat{\eta} = \alpha\mathbf{\Phi}\frac{1}{\alpha}\eta = \mathbf{\Phi}\eta = \mathbf{v}$$

The factor α is cancelled out in the re-transformation and we see that the true solution for eigenvector 2 is identical to the one for eigenvector 1. The choice of eigenvectors is immaterial.

Advantage of Modal Decomposition: Reduction of DOFs

Original space:

The number of dofs is given by the discretization. There can be thousands of dofs, depending on how many elements we chose. So the number of dofs is not a physical quantity, but an artefact of our numerical model.

Structural behaviour:

The true vibration of a structure in nature does not depend on a numerical model – a bridge has not gone to university and knows nothing about finite elements. It just vibrates. The displacement state can be synthesised as a superposition of all mode shapes. The physical reality is such that higher mode shapes contribute less and less to the total response.

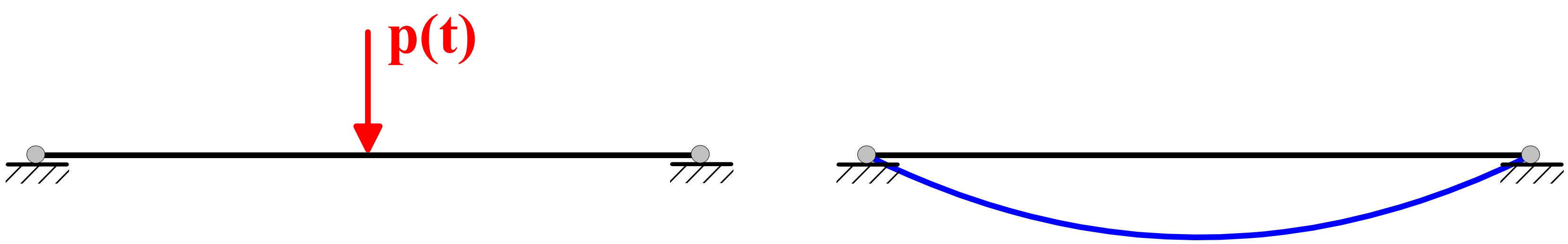
Modal space:

The number of dofs is given by the number of relevant mode shapes. Their number is only a fraction of the number of system dofs. Only some dozens of modes are needed. This leads to substantial savings in computing time.

Extreme Example for the Discrepancy between Original and Modal DOFs

Our example consists in a simply supported beam subjected to a concentrated load in mid-span. We can discretize the problem with finite elements to an arbitrary accuracy so that the number of dofs can be as low as only 10 or as high as 10000. In Dynamics II we will address the question of how to find a suitable discretization.

beam under load in mid-span



The continuous model contains an infinity of mode shapes. Mathematically they are sine waves with increasing wave numbers. The 1st mode shown here has one half-wave, the 2nd has two half-waves, the 3rd three, and so on.

Solution with One Modal DOF

Due to the symmetry of the load only modes with an odd number of half-waves can be excited. How many of them are excited to which degree depends on the *dynamic amplification* of each mode which in turn depends on the ratio of the *load frequencies* to the *eigenfrequencies*.

Now we assume a harmonic loading, i.e. a *mono-harmonic excitation*. We further assume that the load frequency lies near the 1st eigenfrequency. Then the 1st mode is excited in resonance and dominates the entire response – we have a largely *mono-harmonic response* with only then 1st mode being present.

$$w(x,t) = \eta_1(t)\Phi_1(x)$$

In modal space this problem has only one degree of freedom!

Summary

We have developed a method for computing the response of MDOF systems via a superposition of eigenmodes. This method has advantages and disadvantages:

(A) Advantages:

It is a *very efficient procedure*. The computing time can be measured in seconds or minutes, while a direct solution might take hours to run.

We have full control over the damping of *each mode*. We will later see that we lose this control when using direct methods.

(B) Disadvantages:

It is restricted to *linear problems* since the *superposition principle* must hold. Therefore it is not possible to solve problems with *large displacements* and/or *inelastic material behaviour*.

We must have *modal damping*. This can only model distributed damping, *discrete damping elements* cannot be taken into account.

Outlook

The modal superposition method is mainly used in numerical analyses – a manual use is restricted to very simple problems. So in part B of this lecture we will study some numerical examples to become familiar with its capabilities, but also with its problems and limitations.

The limitations of a method always provoke the question if there could not be developed another method which is not subjected to these limitations. The new method would maybe exhibit other weaknesses, but a spectrum of methods with non-overlapping strengths/weaknesses would enable us to chose the optimum algorithm for a given problem or use different methods to capture different effects.

Such alternative methods exist. They can be subsumed under the name of *direct time integration algorithms*. DTI, as it is abbreviated, will be treated in the next lecture.